Equilibrium states of C*-algebras from number theory

Part 1: Quantum statistical mechanical systems and KMS states: introduction and examples

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4 November 2019

C-dynamical system* (A, σ)

- A = C*-algebra; *observables* = self adjoint elements of A
- $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$; the *dynamics* or time evolution on A:

$\sigma_0 = \mathrm{id}$, $\sigma_s \circ \sigma_t = \sigma_{s+t}$ and $t \mapsto \sigma_t(a)$ is norm continuous.

A *state* is a linear functional $\varphi : A \to \mathbb{C}$ such that

$$\varphi(a^*a) \geqslant 0 \quad \text{and} \quad \|\varphi\| = 1 \quad (= \varphi(1) \text{ if } 1 \in A)$$

$\varphi(\sigma_t(a))$ is the *expectation value* of the observable $a \in A^{sa}$ at time $t \in \mathbb{R}$ when the system is in the fixed state φ. (Heisenberg picture)

Basic facts about states:

- If $A = C_0(\Omega_A)$ every probability μ on Ω_A gives a state φ_μ

$$\varphi_\mu(f) = \int_{\Omega_A} f \, d\mu \qquad \text{(all states on } C_0(\Omega_A) \text{ are like this)}$$

- If $A \subset B(\mathcal{H})$, every unit vector $\xi \in \mathcal{H}$ gives a state φ_ξ

$$\varphi_\xi(a) := \langle a\xi, \xi \rangle \qquad \text{(not all are quite like this but, ...)}$$

- GNS construction: for every state φ of A there exist
  - a Hilbert space $\mathcal{H}_\varphi$,
  - a representation $\pi_\varphi : A \to B(\mathcal{H}_\varphi)$, and
  - a cyclic unit vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle.$$

Finite quantum systems (cf. N. Hugenholtz, *C*-algebras and statistical mechanics*, Kingston, 1981)

- $A = \mathrm{Mat}_n(\mathbb{C})$, observables = selfadjoint $n \times n$ matrices.
- Every dynamics σ on $\mathrm{Mat}_n(\mathbb{C})$ arises from a *Hamiltonian* $H = H^* \in \mathrm{Mat}_n(\mathbb{C})$ via

$$\sigma_t(a) := e^{itH} a e^{-itH} \qquad a \in \mathrm{Mat}_n(\mathbb{C}),\ t \in \mathbb{R}.$$

  H is determined up to an additive constant.
- Every state φ of $\mathrm{Mat}_n(\mathbb{C})$ arises from a *density matrix Q* $Q \geqslant 0$; $\mathrm{Tr}\, Q = 1$, via

$$\varphi(a) = \mathrm{Tr}(aQ) \qquad a \in \mathrm{Mat}_n(\mathbb{C}).$$

  The correspondence $\varphi \mapsto Q_\varphi$ is an isomorphism.
- φ is pure iff Q_φ is a rank-one projection.

Finite quantum systems: stationary states, entropy

- A state φ_Q is stationary (i.e. σ-invariant) if

$$\mathrm{Tr}(e^{itH} a e^{-itH} Q) = \mathrm{Tr}(a\, Q) \qquad a \in \mathrm{Mat}_n(\mathbb{C}),\ t \in \mathbb{R},$$

  Since this means $\mathrm{Tr}(a\ e^{-itH} Q e^{itH}) = \mathrm{Tr}(a\, Q)$ for every a,
  φ is stationary $\iff$ $e^{-itH} Q e^{itH} = Q$ $\iff$ $QH = HQ$.
- φ_Q is a pure stationary state iff $Q =$ projection onto a one-dimensional eigenspace of H.
- The *von Neumann entropy* of a state is defined by

$$S(\varphi) := -\,\mathrm{Tr}(Q_\varphi \log Q_\varphi)$$

  Then $0 \leqslant S(\varphi) \leqslant \log n$, and

$$S(\varphi) \text{ is } \begin{cases} 0 \text{ (minimal)} & \text{when } \varphi \text{ is pure} \\ \log n \text{ (maximal)} & \text{when } \varphi = \text{normalized trace.} \end{cases}$$

"Pure states have maximal information; the normalized trace, minimal"

Finite quantum systems: variational principle for equilibrium

Let H be a Hamiltonian in $\mathrm{Mat}_n(\mathbb{C})$, and let φ be a state.
The *free energy* of φ at inverse temperature $\beta = 1/T$ is

$$F(\varphi) := -S(\varphi) + \beta\varphi(H),$$

The *Gibbs state φ_G* is the state with density

$$Q_G := \frac{1}{\mathrm{Tr}(e^{-\beta H})} e^{-\beta H}.$$

The partition function associated to H is $\beta \mapsto \mathrm{Tr}(e^{-\beta H})$.

Variational Principle

The Gibbs state is the unique state minimizing the free energy:

- $F(\varphi) \geqslant -\log \mathrm{Tr}(e^{-\beta H})$;
- $F(\varphi) = -\log \mathrm{Tr}(e^{-\beta H}) \iff \varphi = \varphi_G$.

Finite quantum systems: characterization of Gibbs state

The Gibbs state φ_G is the unique state on $\mathrm{Mat}_n(\mathbb{C})$ satisfying

$$\varphi(ab) = \varphi(b\,\sigma_{i\beta}(a)) \qquad a, b \in \mathrm{Mat}_n(\mathbb{C}), \tag{KMS}$$

where $\sigma_{i\beta}(a) := e^{-\beta H} a e^{\beta H}$.

The proof is an exercise in linear algebra: the Gibbs density $Q_G := \frac{1}{\mathrm{Tr}(e^{-\beta H})} e^{-\beta H}$ is the unique density satisfying

$$\mathrm{Tr}(abQ) = \mathrm{Tr}(be^{-\beta H} a e^{\beta H} Q) \qquad a, b \in \mathrm{Mat}_n(\mathbb{C}).$$

For finite (and other) systems, the KMS condition above is equivalent to the usual equilibrium condition defined in terms of minimal free energy. [HHW] eventually proposed the KMS condition as *defining* equilibrium for general (A, σ).

KMS condition for general (A, σ)

Definition (Haag-Hugenholtz-Winnink, 1967)

A state φ on A satisfies the Kubo-Martin-Schwinger (KMS) condition with respect to σ at inverse temperature $\beta \neq 0$ (φ is a σ-KMS$_\beta$ state), if

$$\varphi(ab) = \varphi(b\,\sigma_{i\beta}(a)) \qquad a, b \in A, \quad \text{with } a \ \sigma\text{-analytic.}$$

Recall:

- $a \in A$ is *σ-analytic* if $t \mapsto \sigma_t(a) \in A$ extends to an A-valued entire function $z \mapsto \sigma_z(a) \in A$.
- The σ-analytic elements form a dense *-subalgebra of A.
- More symmetric, and equivalent, is the condition

$$\varphi(ab) = \varphi(\sigma_{-i\beta/2}(b)\sigma_{i\beta/2}(a)).$$

KMS condition: original (equivalent) formulation

The original KMS condition is closer to the boundary condition for Green functions used by Kubo:

(For $\beta > 0$.)The state φ is KMS$_\beta$ for σ if for any $a, b \in A$ there exists a continuous function

$$f : \{z \in \mathbb{C} \mid 0 \leqslant \operatorname{Im} z \leqslant \beta\} \to \mathbb{C}$$

that is analytic in the open strip $0 < \operatorname{Im} z < \beta$ and satisfies

$$f(t) = \varphi(b\sigma_t(a)), \quad f(t + i\beta) = \varphi(\sigma_t(a)b) \text{ for all } t \in \mathbb{R}.$$

This has the advantage of not relying on analytic elements.

Properties of KMS states

KMS states have the properties expected from equilibrium, e.g.

- Stability
- Passivity
- Minimality

KMS states are intrinsically related to the Tomita-Takesaki theory in von Neumann algebras.

The KMS condition is an essentially *noncommutative* phenomenon:

Proposition

If A is commutative and has a faithful σ-KMS$_\beta$ state for $\beta \neq 0$, then σ is trivial.

Sample proof: KMS $\implies$ stationary

Proposition

Suppose φ is a KMS_β state and $\beta \neq 0$. Then φ is σ-invariant.

Proof when $1 \in A$: Let $b = 1$, and let $a \in A$ be analytic. Then $\sigma_z(a)$ is analytic and

$$\varphi(\sigma_z(a)1) = \varphi(1\sigma_{z+i\beta}(a)),$$

so the entire function $z \mapsto \varphi(\sigma_z(a))$ has period $(i\beta)$; since $\|\phi(\sigma_t(a))\| \leqslant \|a\|$ for $t \in \mathbb{R}$, it is also bounded on $\mathbb{C}$, hence it is constant.

Caveat: the converse is not true, even for finite systems "equilibrium" is strictly stronger than "invariant".

If $\beta = 0$ the KMS_0 condition says φ is a trace (no reference to σ). It is common to *require σ-invariance as part of the definition*, so

$$(\mathrm{KMS}_0 \text{ state}) \iff (\sigma\text{-invariant trace}).$$

The set K_β of KMS$_\beta$-states

- If $\varphi_i \in K_{\beta_i}$, $\beta_i \to \beta$, and $\varphi_i \xrightarrow{w*} \varphi$ then $\varphi \in K_\beta$;
- if $\varphi \in K_\beta$ then the normal extension $\bar{\varphi}$ of φ to $\pi_\varphi(A)''$ is faithful and $\sigma_t^{\bar{\varphi}} \circ \pi_\varphi = \pi_\varphi \circ \sigma_{-\beta t}$;
- in particular, for $\beta \neq 0$ a state φ with faithful GNS-representation can be a σ-KMS$_\beta$-state for at most one dynamics σ, and then if such a nontrivial dynamics σ is fixed, β is also uniquely determined;
- if A is separable, unital, then K_β is a Choquet simplex in the state space of A, (i.e., K_β is weak*-closed convex and every $\varphi \in K_\beta$ is the barycenter of a unique probability measure supported on $Extr(K_\beta)$);
- $\varphi \in K_\beta$ is extremal (a pure phase) iff $\pi_\varphi(A)''$ is a factor

Phase transition and symmetry breaking

Phase transition is a change in the physical properties of a system.

Example: transition between the solid, liquid, and gaseous phases.

Phase transitions often (but not always) involve phases with different symmetry. Some intuitive examples are:

- A snowflake is less symmetric than a spherical drop of water.
- Ferromagnets are capable of spontaneous magnetization as magnetic dipoles "align" coherently at low temperatures.

In C*-algebraic terms there are two interpretations:

$$\begin{cases} K_\beta \text{ is not a singleton at a given } \beta \text{ (Sakai)} \\ \text{the nature of } K_\beta \text{ changes as } \beta \text{ goes through a critical value} \end{cases}$$

Spontaneous symmetry breaking occurs when the symmetries of K_β change as β changes. Typically (but not necessarily) the symmetry group of K_β becomes smaller as the inverse temperature β increases.

Periodic dynamics on the Toeplitz algebra

- $A = \mathcal{T} :=$ universal C*-algebra of an isometry S;
- $\sigma :=$ periodic dynamics determined by $\sigma_t(S) = e^{it}S$;
- $\{S^m S^{*n} : m, n \geqslant 0\}$ spans a dense *-subalgebra; and

$$\sigma_t(S^m S^{*n}) = e^{i(m-n)t} S^m S^{*n}$$

- spanning elements are analytic, and the KMS$_\beta$ condition implies

$$\varphi(S^m S^{*n}) = e^{-m\beta}\varphi(S^{*n}S^m) = e^{-(m-n)\beta}\varphi(S^m S^{*n})$$

- For each β, there is at most one KMS$_\beta$ state; it is given by

$$\varphi(S^m S^{*n}) = \begin{cases} 0 & \text{for } m \neq n \\ e^{-n\beta} & \text{for } m = n. \end{cases}$$

$\exists$ a KMS$_\beta$ state of $(\mathcal{T}, \sigma)$: "dynamical system proof"

A unique KMS$_\beta$ state does exist for each $\beta \geqslant 0$.

- $\mathcal{T} \cong c \rtimes \mathbb{N}$ where $c \cong \overline{\text{span}}\{S^n S^{*n} : n \in \mathbb{N}\}$
- Since $\varphi(S^m S^{*n}) = 0$ for $m \neq n$, a KMS state factors through the conditional expectation $\Phi : c \rtimes \mathbb{N} \to c$ and is determined by a probability measure μ_β on $\hat{c} = \mathbb{N} \sqcup \{\infty\}$.
- Since $\varphi(S^n S^{*n}) = e^{-\beta n}$ the measure μ_β must satisfy

$$\mu_\beta(\{n\}) = \varphi(S^n S^{*n}) - \varphi(S^{n+1} S^{*n+1}) = e^{-\beta n} - e^{-\beta(n+1)}.$$

- So choose μ_β to be geometric: $\mu_\beta(\{n\}) := (1 - e^{-\beta})e^{-\beta n}$.
- Then the state of $\mathcal{T}$ induced through Φ satisfies

$$\varphi_\beta(S^m S^{*n}) = \begin{cases} 0 & \text{for } m \neq n \\ e^{-\beta n} & \text{for } m = n. \end{cases}$$

KMS states on the Toeplitz algebra: "Hilbert space proof"

A unique KMS$_\beta$ state does exist for each $\beta \geqslant 0$.

- $\mathcal{T} \cong$ C*-algebra of the unilateral shift $S : \delta_n \mapsto \delta_{n+1}$ on $\ell^2(\mathbb{N})$
- Define $H : \delta_n \mapsto n\delta_n$ on $\ell^2(\mathbb{N})$, then
  - the dynamics is given spatially by $\sigma_t = \mathrm{Ad}_{e^{itH}}$, and
  - $e^{-\beta H}$ is trace class for $\beta > 0$.
- $Z_\beta := \mathrm{Tr}(e^{-\beta H}) = \sum_{n=0}^{\infty} e^{-n\beta} = \frac{1}{1-e^{-\beta}}$.
- Define a 'generalized Gibbs state' by $\varphi_\beta(x) := \frac{1}{Z_\beta}\mathrm{Tr}(x\, e^{-\beta H})$.
- Then φ_β is σ-KMS$_\beta$ state, and

$$\varphi_\beta(S^m S^{*n}) = \frac{1}{Z_\beta}\sum_k \langle S^m S^{*n} e^{-\beta k}\delta_k, \delta_k\rangle = \begin{cases} 0 & \text{for } m \neq n \\ e^{-n\beta} & \text{for } m = n. \end{cases}$$

- If $\beta = 0$, there is a unique σ-invariant trace; the weak*-$\lim_{\beta\to 0^+} \varphi_\beta$.

In the late 80's Bernard Julia and independently Donald Spector proposed an interpretation of the Riemann zeta function as the partition function of a quantum system. For each prime number p there is a particle with creation operator $|p\rangle$ and energy $\log p$. The partition function of the single particle system is

$$Z_p(\beta) = \sum_{k=0}^{\infty} e^{-k\beta \log p} = \frac{1}{1 - p^{-\beta}}.$$

Assuming the prime numbers behave like bosons and have a common vacuum vector, the system consisting of all primes has partition function equal to the Euler product form of the Riemann zeta function.

$$Z(\beta) = \prod_p \frac{1}{1 - p^{-\beta}} = \zeta(\beta)$$

A C*-algebra for the Riemann gas

For each p the creation operator $|p\rangle$ is an isometry $s_p : \delta_{p^k} \mapsto \delta_{p^{k+1}}$ acting on $\ell^2(p^{\mathbb{N}}) = \overline{\text{span}}\{|p\rangle^k \phi : k = 0, 1, 2, \dots\}$, generating a copy $\mathcal{T}_p$ of $\mathcal{T}$.

Taken together, these bosonic 'primons' give a tensor product:

$$\bigotimes_p \mathcal{T}_p \cong \mathcal{T}(\mathbb{N}^\times)$$

generated by isometries $\otimes_p s_p^{k_p} \cong L_n$ (with $n = \prod_p p^{k_p}$)

acting on $\otimes' \ell^2(p^{\mathbb{N}}) \cong \ell^2(\prod_p' p^{\mathbb{N}}) \cong \ell^2(\mathbb{N}^\times)$

with the tensor product dynamics $\otimes \sigma_t^p \cong \sigma_t$ given by

$$\sigma_t(L_m L_n^*) = \left(\tfrac{m}{n}\right)^{it} L_m L_n^* = e^{it \log m/n} L_m L_n^* e^{-it \log m/n}.$$

For each $\beta \geqslant 0$, the product state $\otimes \varphi_{p,\beta}$ is the unique KMS$_\beta$ state but the partition function is $\text{Tr}(e^{-\beta H}) = \zeta(\beta)$, which has a pole at $\beta = 1$.

Bost and Connes pointed out that the Riemann gas had no interaction and considered the **Hecke pair**

$$P_{\mathbb{Z}}^{+} := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & \mathbb{Q} \\ 0 & \mathbb{Q}_{+}^{*} \end{pmatrix} =: P_{\mathbb{Q}}^{+}$$

Definition

The *Hecke C*-algebra* of Bost and Connes is the C*-algebra $\mathcal{C}_{\mathbb{Q}}$ generated by the characteristic functions of double cosets $[\gamma] \in P_{\mathbb{Z}}^{+} \backslash P_{\mathbb{Q}}^{+} / P_{\mathbb{Z}}^{+}$ acting on $\ell^2(P_{\mathbb{Z}}^{+} \backslash P_{\mathbb{Q}}^{+})$ by convolution:

$$(f * g)(\gamma) := \sum_{\gamma_1 \in P_{\mathbb{Z}}^{+} \backslash P_{\mathbb{Q}}^{+}} f(\gamma \gamma_1^{-1}) g(\gamma_1)$$

The addition and multiplication of numbers are both incorporated into this construction.

The Bost–Connes C*-algebra as semigroup crossed product

Alternative description in terms of the ring of integral adeles $\hat{\mathbb{Z}} := \prod_p \mathbb{Z}_p$.

$$\mathcal{C}_{\mathbb{Q}} \cong C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times = \overline{\text{span}}\{\mu_m f \mu_n^* : m, n \in \mathbb{N}^\times,\ f \in C(\hat{\mathbb{Z}})\}$$

$$\sigma_t(\mu_m f \mu_n^*) = (m/n)^{it} \mu_m f \mu_n^*$$

For each unit $\mathbf{u} \in \hat{\mathbb{Z}}^*$ there is an irreducible representation

$$\pi_{\mathbf{u}} : C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times \to \mathcal{B}(\ell^2(\mathbb{N}^\times))$$

$$\pi_{\mathbf{u}}(f)\delta_n = f(n \cdot \mathbf{u})\delta_n \qquad \pi_{\mathbf{u}}(\mu_n) = L_n$$

As before, let $H\delta_n = (\log n)\delta_n$, so that $\sigma_t \sim \text{Ad}_{e^{itH}}$ and $\text{Tr}(e^{-\beta H}) = \zeta(\beta)$
Then the generalized Gibbs state

$$\omega_{\beta,\mathbf{u}}(\cdot) := \frac{1}{\zeta(\beta)} \text{Tr}(\pi_{\mathbf{u}}(\cdot) e^{-\beta H})$$

is a KMS_β state for $\beta > 1$.

Theorem (Bost–Connes, '95)

1. *For each* $0 < \beta \leqslant 1$ *there is a unique* KMS$_\beta$ *state of* $(\mathcal{C}_\mathbb{Q}, \sigma)$. *It is an injective type* III$_1$ *factor state, invariant under the action of* $\operatorname{Aut}\mathbb{Q}/\mathbb{Z}$.
2. *For each* $1 < \beta \leqslant \infty$ *the extremal* KMS$_\beta$ *states* $\phi_{\beta,\chi}$ *are parametrized by the complex embeddings* $\chi : \mathbb{Q}^{cycl} \to \mathbb{C}$ *of the maximal cyclotomic extension of* $\mathbb{Q}$. *These are type I factor states, on which the action of* $\mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \cong \operatorname{Aut}\mathbb{Q}/\mathbb{Z}$ *is free and transitive.*
3. *The partition function of the system is the Riemann zeta function.*

Note: H, hence $\mathrm{Tr}(e^{-\beta H})$ does not depend on $\mathbf{u}$
There are group isomorphisms

$$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \cong \operatorname{Aut}\mathbb{Q}/\mathbb{Z} \cong \hat{\mathbb{Z}}^*$$

$$\chi \mapsto \mathbf{u}$$

A phase transition on $(C_r^*(R \rtimes R^\times), \sigma^N)$

Consider the semidirect product $\mathbb{N} \rtimes \mathbb{N}^\times$ or, more generally, the "$ax + b$ semigroup" $R \rtimes R^\times$ of the ring of integers in an algebraic number field.

$C_r^*(R \rtimes R^\times)$ Toeplitz-type C*-algebra generated by isometries:

$$T_{(b,a)}\delta_{(x,y)} = \delta_{(b+ax,ay)} \quad \text{acting on } \ell^2(R \rtimes R^\times),$$

with dynamics $\sigma_t(T_{(b,a)}) = [R : aR]^{it} T_{(b,a)}, \quad t \in \mathbb{R}.$

Theorem (Cuntz–Deninger–L, '13; cf. L–Raeburn, '10)

For $\beta > 2$ the KMS$_\beta$ states of $(C_r^*(R \rtimes R^\times), \sigma)$ are affinely isomorphic to the tracial states of

$$\mathcal{A} := \bigoplus_{\gamma \in \mathcal{C}\ell_K} C^*(J_\gamma \rtimes U_K)$$

with J_γ an integral ideal representing the ideal class $\gamma \in \mathcal{C}\ell_K$.