P: left cancellative semigroup. For $p \in P$, define $V_p : \ell^2 P \to \ell^2 P$, $\delta_x \mapsto \delta_{px}$. δ_x: delta function in x. $C^*_\lambda(P) := C^*(\{V_p \colon p \in P\}) \subseteq \mathcal{L}(\ell^2 P)$. This gen. $C^*_\lambda(G)$.

Examples

1. positive cones (left order: left-invariant total order)

first in totally ordered ab. groups, e.g. $\mathbb{N} \subseteq \mathbb{Z}$: $C^*_\lambda(\mathbb{N})$ Toeplitz algebra, C*(unilateral shift) [Coburn]

For $G \subseteq (\mathbb{R}, +)$, $P = G \cap [0, \infty)$ [Douglas, ...] C* completely determines P.

2. (Right-angled) Artin monoids [Nica, Crisp-Laca, ...]

e.g. $\mathbb{N} \times \mathbb{N}$ [Picture: graph]: $C^*_\lambda(\mathbb{N} \times \mathbb{N}) \cong C^*_\lambda(\mathbb{N}) \otimes C^*_\lambda(\mathbb{N})$

e.g. $\mathbb{N} * \mathbb{N}$ [Picture: graph]: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$, where $\mathcal{T}_2$ is the Toeplitz extension of the Cuntz algebra $\mathcal{O}_2$, $0 \to \mathcal{K} \to \mathcal{T}_2 \to \mathcal{O}_2 \to 0$

$\Gamma = (V, E)$ graph, $E \subseteq V \times V$. $A^+_\Gamma := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \, \forall \, (v, w) \in E \rangle^+$; embeds into RAAG

gen. graph products [some aspects discussed in Book]

gen. Artin monoids (e.g. Braid monoids, e.g. $B_3^+ = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle^+$) [interesting to study]

Baumslag-Solitar monoid $\big\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a}\mathbf{b}^{\mathbf{k}} = \mathbf{b}^{\mathbf{l}}\mathbf{a} \big\rangle^+$ or $\big\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{b}^{\mathbf{l}}\mathbf{a}\mathbf{b}^{\mathbf{k}} \big\rangle^+$ [Spielberg], gen. graphs of sgps $\longrightarrow$ Cheng

Thompson monoid $\mathbf{F}^+ = \langle \mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \ldots \mid \mathbf{x_n}\mathbf{x_k} = \mathbf{x_k}\mathbf{x_{n+1}} \text{ for } \mathbf{k} < \mathbf{n} \rangle^+$

General machinery to find interesting semigroups given by positive presentations?

3. $\mathbf{R}^\times$ or $\mathbf{R} \rtimes \mathbf{R}^\times$, R: integral domain

$R^\times = R \setminus \{0\}$, $R \rtimes R^\times = R \times R^\times$ as sets, multiplication given by $(d, c)(b, a) = (d + cb, ca)$.

e.g. R ring of alg. int. in number field K; $\mathbb{Z}[i]$ or $\mathbb{Z}[\zeta]$ (ζ root of unity)

4. $\mathbf{P} \subseteq \mathbb{Z}^{\mathbf{n}}$ **fin. gen.** e.g. $P \subseteq \mathbb{Z}$: numerical semigroup, e.g. $P = \mathbb{N} \setminus \{1\} = \{0, 2, 3, 4, \ldots\}$.

Analyse structure of $C^*_\lambda(P)$: ideal structure, nuclearity, K-theory $\longrightarrow$ classification / rigidity (reconstruction of P?) — focus: K-theory; tool: crossed product descriptions and GPD models

[will assume basic knowledge of K-theory]

Reference: book & survey "Semigroup C*-algebras" on the arXiv

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Assume $e \in P \subseteq G$. Let I_l be the semigroup generated by $\{V_p\colon p \in P\}$ and $\{V_q^*\colon q \in P\}$. Elements in I_l are partial isometries on $\ell^2 P$. Write $I_l^\times := I_l \setminus \{0\}$. For every $V \in I_l^\times$, there exists a unique $g \in G$ such that $V\delta_x = 0$ or $V\delta_x = \delta_{gx}$ for all $x \in P$. Define $\sigma(V) := g$. We have $\sigma(V^*) = g^{-1}$ and $\sigma(V_1 V_2) = \sigma(V_1)\sigma(V_2)$ if $V_1 V_2 \neq 0$. Note: $V = \lambda_g V^* V$. Let $D := \overline{\operatorname{span}}\{V \in I_l^\times\colon \sigma(V) = e\} \subseteq C_\lambda^*(P)$. Claim: $D = C_\lambda^*(P) \cap \ell^\infty(P)$. "$\subseteq$" OK. "$\supseteq$": $\vartheta : \mathcal{L}(\ell^2 P) \to \ell^\infty(P)$ faithful conditional expectation given by $\langle \vartheta(T)\delta_x, \delta_y \rangle = \delta_{x,y} \langle T\delta_x, \delta_x \rangle$ for all $x, y \in P$. We have for all $V \in I_l^\times$: $\vartheta(V) = V$ if $\sigma(V) = e$ and $\vartheta(V) = 0$ if $\sigma(V) \neq e$. Hence $D \supseteq \vartheta(C_\lambda^*(P)) \supseteq C_\lambda^*(P) \cap \ell^\infty(P)$. Note: We obtain faithful conditional expectation $\vartheta : C_\lambda^*(P) \twoheadrightarrow D$.

We want to define a partial action $G \curvearrowright D$ such that $C_\lambda^*(P) \cong D \rtimes_r G$. Define

$$D_{g^{-1}} := \overline{\operatorname{span}}\{V^*V\colon V \in I_l^\times,\, \sigma(V) = g\},$$

$\alpha_g^* : D_{g^{-1}} \to D_g$, $V^*V \mapsto VV^* = \lambda_g VV^*VV^*\lambda_g^* = \lambda_g VV^*\lambda_g^*$. Dual action on $\Omega := \operatorname{Spec} D$: $U_{g^{-1}} := \operatorname{Spec} D_{g^{-1}} = \{\chi \in \Omega\colon \exists V \in I_l^\times, \sigma(V) = g : \chi(V^*V) = 1\}$, $\alpha_g : U_{g^{-1}} \to U_g$, $\chi \mapsto \chi \circ \alpha_{g^{-1}}^*$. Write $g.x = \alpha_g(x)$. Let us construct $D \rtimes_r G$. $M : D = C(\Omega) \to \mathcal{L}(\ell^2\Omega)$, $M(f)\xi = f \cdot \xi$. For $g \in G$, define $M_g : D \to \mathcal{L}(\ell^2\Omega)$, $M_g(f)\xi := f|_{U_g}(g.\sqcup) \cdot \xi|_{U_{g^{-1}}}$ (on $U_{g^{-1}}$, $x \mapsto g.x$ is defined). Let $H := \ell^2 G \otimes \ell^2\Omega$. Define $\mu : D \to \mathcal{L}(H)$, $\mu(f)(\delta_h \otimes \xi) := \delta_h \otimes M_h(f)\xi$. Let E_g be the orth. proj. onto $\overline{\mu(D_{g^{-1}})(H)} = \bigoplus_h \delta_h \otimes \ell^2(U_{h^{-1}} \cap U_{(gh)^{-1}})$. [$M_h(f)\delta_\chi = f|_{U_h}(h.\chi)\delta_\chi$ if $\chi \in U_{h^{-1}}$, otherwise 0; so if $f \in D_{g^{-1}} = C_0(U_{g^{-1}})$, then $h.\chi \in U_{g^{-1}} \Rightarrow \chi \in h^{-1}.U_{g^{-1}}$ if non-zero, so $\chi \in U_{h^{-1}} \cap h^{-1}.U_{g^{-1}} = h^{-1}.(U_h \cap U_{g^{-1}}) = U_{h^{-1}} \cap U_{(gh)^{-1}}$] Define $W_g := (\lambda_g \otimes I)E_g$. Point: W_g is partial isometry with $W_g f W_g^* = f(g.\sqcup)$ if $f \in C_0(U_{g^{-1}})$. Then $D \rtimes_r G := \overline{\operatorname{span}}\{\mu(f)W_g\colon f \in C_0(U_g) = D_g,\, g \in G\}$. Faithful conditional expectation θ on $\mathcal{L}(H) = \mathcal{L}(\ell^2 G \otimes \ell^2\Omega)$ given by $\langle \theta(T)(\delta_h \otimes \delta_\chi), \delta_{h'} \otimes \delta_{\chi'} \rangle = \delta_{h,h'} \langle T(\delta_h \otimes \delta_\chi), \delta_{h'} \otimes \delta_{\chi'} \rangle$. Restriction gives faithful conditional expectation $\theta : D \rtimes_r G \twoheadrightarrow D$.

$\omega : D \to \mathbb{C}$, $\omega(V) = 1$ if $V = I_{\ell^2 P}$ and 0 else, for $V \in I_l^\times$ with $\sigma(V) = e$. $\omega = \langle \sqcup\delta_e, \delta_e \rangle$. $P \hookrightarrow G \times \Omega$, $x \mapsto (x, \omega)$ gives $\ell^2 P \hookrightarrow \ell^2 G \otimes \ell^2\Omega = H$, $\delta_x \mapsto \delta_x \otimes \delta_\omega$. Claim: $\ell^2 P$ is $D \rtimes_r G$ invariant and for $T \in D \rtimes_r G$, $T|_{\ell^2 P} : \ell^2 P \to \ell^2 P$ lies in $C_\lambda^*(P)$: $\mu(f)(\delta_x \otimes \delta_\omega) = \delta_x \otimes f|_{U_x}(x.\omega)\delta_\omega$ if $\omega \in U_{x^{-1}}$ and 0 else. Want: $\mu(V)(\delta_x \otimes \delta_\omega) = V(\delta_x) \otimes \delta_\omega$ for all $V \in I_l^\times$ with $\sigma(V) = e$. $V|_{U_x}(x.\omega) = (x.\omega)(VV_xV_x^*) = \omega(\lambda_x^{-1} VV_xV_x^*\lambda_x) = 1$ if $V \geq V_xV_x^*$ and 0 else. $V\delta_x = 1$ iff $V\delta_{xy} = 1 \ \forall\, y \in P$ iff $V \geq V_xV_x^*$. OK. $(\lambda_g \otimes E_g)(\delta_x \otimes \delta_\omega) \neq 0$ iff $\omega \in U_{(gx)^{-1}}$ iff $\exists\ V \in I_l^\times$ with $\sigma(V) = gx$ with $V^*V = 1$ iff $0 \neq V\delta_e$ & $V\delta_e = \delta_{gx}$ iff $gx \in P$. So $W_g(\delta_x \otimes \delta_\omega) = \delta_{gx} \otimes \delta_\omega$ if $gx \in P$ and 0 otherwise. Hence for typical element $\mu(VV^*)W_g$, $\sigma(V) = g$, $\mu(VV^*)W_g(\delta_x \otimes \delta_\omega) = (VV^*\lambda_g)(\delta_x) \otimes \delta_\omega = V(\delta_x) \otimes \delta_\omega$. OK. In particular, $W_p(\delta_x \otimes \delta_\omega) = V_p(\delta_x) \otimes \delta_\omega$.

Hence $D \rtimes_r G \to C_\lambda^*(P)$, $T \mapsto T|_{\ell^2 P}$ is well-defined and surjective, $W_p \mapsto V_p$, id on D. CD with θ and ϑ faithful $\Rightarrow$ injective. Hence: $C_\lambda^*(P) \cong D \rtimes_r G \cong C_r^*(G \ltimes \Omega)$.

Use this to study ideal structure, nuclearity ... Can use this to define full semigroup C*. Alternative: Look at $C_\lambda^*(I_l)$, $C^*(I_l)$, [Paterson]: GPD models. BUT we need a condition to make sure that $C_\lambda^*(P) \cong C_\lambda^*(I_l)$: the independence condition.

General K-theory formula. $G \curvearrowright \Omega$, Ω totally disconnected, second countable. independence: $\exists\, \mathcal{V}$ family of compact open subspaces of Ω s.t.

- $\{1_U \colon U \in \mathcal{V}\}$ generates $C_0(\Omega)$;
- $\mathcal{V} \cup \{\emptyset\}$ is closed under intersections;
- $\mathcal{V}$ is G-invariant;
- $\forall\, U, U_1, \dots, U_n \in \mathcal{V}$, $U = \bigcup_{i=1}^n U_i \Rightarrow U = U_i$ for some i.

Examples: If G is trival, then $\mathcal{V}$ always exists. Full shift $G \curvearrowright \{0, \dots, n\}^G$. For $F \subseteq G$, let $\pi_F : \{0, \dots, n\}^G \twoheadrightarrow \{0, \dots, n\}^F$ be can. proj. $\mathcal{V} = \left\{\pi_F^{-1}(\varphi) \colon F \subseteq G \text{ finite}, \varphi \in \{1, \dots, n\}^F\right\}$.

Counterexample: Let G act minimally on Ω, i.e. for every non-empty open subset U of Ω, we have $\Omega = \bigcup_{g \in G} gU$. Suppose further that there exists a non-zero G-invariant Borel measure μ on Ω (which holds if G is amenable and Ω is compact). Independence holds if and only if Ω is discrete. Proof: μ has full support by minimality ($\mathrm{supp}(\mu)$ is closed G-invariant). Now assume $\mathcal{V}$ exists. Take $U, V \in \mathcal{V}$. By minimality, we have $V = \bigcup_{g \in G} gU \cap V$. Since V is compact, there must be finitely many g_1, ..., g_n in G such that $V = \bigcup_{i=1}^n g_i U \cap V$ with $g_i U \cap V \neq \emptyset$. By independence of $\mathcal{V}$, we conclude that we must have $V = g_i U \cap V$ for some i (note that $g_i U \cap V$ is again an element in $\mathcal{V}$). Thus $V \subseteq g_i U$. So we have seen that whenever given U, V in $\mathcal{V}$, we can find a group element $t \in G$ such that $V \subseteq tU$. Reversing the roles of V and U, we can also find a group element s with $sU \subseteq V$. So we have $sU \subseteq V \subseteq tU$. Now let us apply our measure μ: We have

$$0 \leq \mu(V \setminus sU) = \mu(V) - \mu(U) = -\mu(tU \setminus V) \leq 0.$$

Hence we conclude that $\mu(V \setminus sU) = 0$. Thus $V = sU$. So we have shown that $\mathcal{V}$ consists of one single G-orbit, i.e. we have $\mathcal{V} = G \cdot U$ for some (hence every) $U \in \mathcal{V}$. Using μ, we see that sets of the form gU are either equal or disjoint. As $\mathcal{V}$ is a generating set for the compact open subsets of Ω, and since these generate the topology of Ω, we conclude that Ω must be discrete.

Theorem: Suppose $G \curvearrowright \Omega$ satisfies independence, G satisfies BCwC. Then

$$K_*(C_0(\Omega) \rtimes_r G) \cong \bigoplus_{[U] \in G \backslash \mathcal{V}^\times} K_*(C^*_\lambda(G_U)), \text{ where } G_U = \{g \in G \colon g.U = U\}.$$

Proof uses Going-Down Principle (Chabert, Echterhoff, Meyer, Nest, Oyono-Oyono, ...): $G \curvearrowright A$, $G \curvearrowright B$, G sat. BCwC in A, B, $x \in KK^G(A, B)$. If $x \rtimes F$ induces isomorphisms $K_*(A \rtimes F) \cong K_*(B \rtimes F)$ for all finite subgroups F of G, then $x \rtimes_r G$ induces isomorphism $K_*(A \rtimes_r G) \cong K_*(B \rtimes_r G)$.

View $\mathcal{V}$ as discrete space. Consider $\phi : C_0(\mathcal{V}) \to \mathcal{K}(\ell^2\mathcal{V}) \otimes C_0(\Omega)$, $\delta_U \mapsto e_{U,U} \otimes 1_U$. Note: We need $\mathcal{K}$ because δ are pairwise orthogonal. ϕ G-equivariant. Let $x = [\phi]$ in $KK^G(C_0(\mathcal{V}), \mathcal{K}(\ell^2\mathcal{V}) \otimes C_0(\Omega)) \cong KK^G(C_0(\mathcal{V}), C_0(\Omega))$. To show: $x \rtimes F$ induces K_*-isomorphism for every finite subgroup F of G. Proof for $F = \{e\}$ (general case similar). x induces $K_*(C_0(\mathcal{V})) \cong \bigoplus_{U \in \mathcal{V}} \mathbb{Z}[\delta_U] \to K_*(C_0(\Omega))$, $[\delta_U] \mapsto [1_U]$. This explains construction of ϕ. Show: This is isomorphism. Write $\mathcal{V} = \bigcup_{i=1}^\infty \mathcal{V}_i$, where $V_1 \subseteq V_2 \subseteq \dots$, $|\mathcal{V}_i| < \infty$, $\mathcal{V}_i \cup \{\emptyset\}$ closed under intersections. CD: $K_*(C_0(\mathcal{V})) \cong \varinjlim_i K_*(C_0(\mathcal{V}_i))$ — $K_*(x) = K_*(\phi)$ — $K_*(C_0(\Omega)) \cong \varinjlim_i K_*(C^*(1_{\mathcal{V}_i}))$ & $K_*(\phi_i) : K_*(C_0(\mathcal{V}_i)) \to K_*(C^*(1_{\mathcal{V}_i}))$, $[\delta_U] \mapsto [1_U]$. So it suffices to show that $K_*(\phi_i)$ is an isomorphism. Now use

$$C^*(1_{\mathcal{V}_i}) \cong C_0(\mathcal{V}_i),\ 1_U - 1_{\bigcup_{V \in \mathcal{V}_i,\, V \subsetneq U} V} \mathrel{\hat{=}} \delta_U,\ 1_U \mapsto \sum_{V \in \mathcal{V}_i,\, V \subseteq U} \delta_V$$

Note: $1_U - 1_{\bigcup_{V \in \mathcal{V}_i,\, V \subsetneq U} V} \neq 0$ by independence! Now compose $K_*(\phi_i)$ with this isomorphism in K-theory.

$$\bigoplus_{U \in \mathcal{V}_i} \mathbb{Z}[\delta_U] \cong K_*(C_0(\mathcal{V}_i)) \to K_*(C^*(1_{\mathcal{V}_i})) \cong K_*(C_0(\mathcal{V}_i)) \cong \bigoplus_{U \in \mathcal{V}_i} \mathbb{Z}[\delta_U],\ [\delta_U] \mapsto \sum_{V \in \mathcal{V}_i,\, V \subseteq U} [\delta_V].$$

Write $\mathcal{V}_i = \left\{U^{(1)}, U^{(2)}, \dots\right\}$ such that $\forall\ j,\ k$: $U^{(j)} \subseteq U^{(k)} \Rightarrow j \geq k$. Then the above map becomes the map $\bigoplus_{h=1}^{|\mathcal{V}_i|} \mathbb{Z} \to \bigoplus_{h=1}^{|\mathcal{V}_i|} \mathbb{Z}$ whose matrix has 1 on diagonal, 0 above diagonal $\Rightarrow$ invertible! $\square$

Note: Going-Down Principle is crucial: It allows us to go from infinite G to finite F. Otherwise, we could not orthogonalize in C*.

Let Σ be a finite group and Γ a countable group. Let $\mathrm{con}\,\Sigma$ be the set of conjugacy classes in Σ, and $\mathrm{con}^\times \Sigma := \mathrm{con}\,\Sigma \setminus \{\{1\}\}$ the set of non-trivial conjugacy classes. Let $\mathcal{C}$ be the set of conjugacy classes of finite subgroups of Γ. For a finite subgroup C of Γ, let $F(C)$ be the set of non-empty finite subsets of the right coset space $C\backslash\Gamma$ which are not of the form $\pi^{-1}(Y)$ for a finite subgroup $D \subseteq \Gamma$ with $C \subsetneq D$ and $Y \subseteq D\backslash\Gamma$, where $\pi : C\backslash\Gamma \twoheadrightarrow D\backslash\Gamma$ is the canonical projection. The normalizer $N_C := \left\{\gamma \in \Gamma\colon\ \gamma C \gamma^{-1} = C\right\}$ acts on $F(C)$ by left multiplication, and we denote the set of orbits by $N_C\backslash F(C)$. Given $X \in F(C)$, we write $C \cdot X := \bigsqcup_{x \in X} C \cdot x$ and let $(\mathrm{con}^\times \Sigma)^{C \cdot X}$ be the set of functions $C \cdot X \to \mathrm{con}^\times \Sigma$. $\gamma \in C$ acts on $\varphi \in (\mathrm{con}^\times \Sigma)^{C \cdot X}$ via $(\gamma.\varphi)(x) = \varphi(\gamma^{-1}x)$, and we set $\mathrm{Stab}_C(\varphi) = \{\gamma \in C\colon\ \gamma.\varphi = \varphi\}$ for $\varphi \in (\mathrm{con}^\times \Sigma)^{C \cdot X}$.

Theorem: If Γ satisfies the BCwC, then

$$K_*(C^*_\lambda((\bigoplus_\Gamma \Sigma) \rtimes \Gamma)) \cong K_*(C^*_\lambda(\Gamma)) \oplus \left(\bigoplus_{[C] \in \mathcal{C}} \ \bigoplus_{[X] \in N_C\backslash F(C)} \ \bigoplus_{[\varphi] \in C\backslash((\mathrm{con}^\times \Sigma)^{C \cdot X})} K_*(C^*_\lambda(\mathrm{Stab}_C(\varphi))) \right).$$

Here we take one representative C out of each class in $\mathcal{C}$, one representative X out of each class in $N_C\backslash F(C)$, and one representative φ out of each class in $C\backslash\left((\mathrm{con}^\times \Sigma)^{C \cdot X}\right)$.

K-theory for semigroup C*. Goal: Apply general K_* formula to $C^*_\lambda(P)$. $I_l \cong$ inverse semigroup of partial bijections on P generated by $l_p : P \cong pP \subseteq P$, $x \mapsto px$. $V : (\delta_x \mapsto \delta_{\varphi(x)}) \mathrel{\hat{=}} \varphi$. $V_p \mathrel{\hat{=}} l_p$. $\left\{V \in I_l^\times \colon \sigma(V) = e\right\} \cup \{0\} \mathrel{\hat{=}} \mathcal{J} =$ domains and ranges of partial bijections in I_l.

$e \in P \subseteq G$, G countable. We learned: $C^*_\lambda(P) \cong D \rtimes_r G$. partial crossed product $\longrightarrow$ crossed product for global action? Needed in general K_* formula.

Definition: $P \subseteq G$ satisfies the Toeplitz condition if for every $g \in G$ with $P \cap g^{-1}P \neq \emptyset$, the partial bijection $P \cap g^{-1}P \to gP \cap P$, $x \mapsto gx$ lies in I_l.

Ex: If P is right reversible, i.e., $Pp \cap Pq \neq \emptyset$ for all $p, q \in P$, then $P \subseteq G$ is Toeplitz: In that case, we may assume that G is the group of left quotients of P, i.e., $G = P^{-1}P$. We claim that $P \subseteq G$ is Toeplitz: Take $g \in G$, and write $g = q^{-1}p$ for some $p, q \in P$. Then the partial bijection

$$g^{-1}P \cap P \to P \cap gP,\ x \mapsto gx$$

is the composition of

$$l_q^{-1} : qP \to P,\ qx \mapsto x \text{ and } l_p : P \to pP,\ x \mapsto px.$$

This is because

$$g^{-1}P \cap P = p^{-1}qP \cap P = p^{-1}(qP) \cap P = p^{-1}(\mathrm{dom}(q^{-1})) = \mathrm{dom}(q^{-1}p),$$

and for $x \in g^{-1}P \cap P = \mathrm{dom}(q^{-1}p)$, we have $gx = q^{-1}px = (q^{-1}p)(x)$.

Counterex: The embedding $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2/\mathbb{F}_2''$ is not Toeplitz. Also, we could take the Thompson group

$$F = \langle x_0, x_1, x_2, \dots \mid x_n x_k = x_k x_{n+1} \text{ for } k < n \rangle .$$

The homomorphism $\mathbb{N} * \mathbb{N} \to F$, $a \mapsto x_0$, $b \mapsto x_1$ turns out to be an embedding which is not Toeplitz either. But $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2$, and we just need one Toeplitz embedding, as we can choose any group embedding. Is there a semigroup without any Toeplitz group embedding?

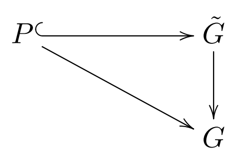
Lemma: $P \subseteq G$ is Toeplitz iff for all $p, q \in P$, there exists a partial bijection $s \in I_l(P)$ with $s(q) = p$ and the intersection $P \cap qp^{-1}P$, taken in G, is contained in the domain $\mathrm{dom}(s)$. Proof: If $g \in G$ satisfies $g^{-1}P \cap P \neq \emptyset$, then there exists p, q in P with $g^{-1}p = q$, i.e., $g = pq^{-1}$. This shows that $P \subseteq G$ is Toeplitz if and only if for all $p, q \in P$, the partial bijection

$$qp^{-1}P \cap P \to P \cap pq^{-1}P,\ x \mapsto pq^{-1}x$$

lies in $I_l(P)$. But this is precisely what our condition says. $\square$

Cor: Suppose that we have a semigroup P with two group embeddings $P \hookrightarrow G$ and $P \hookrightarrow \tilde{G}$. Furthermore, assume that there is a group homomorphism $\tilde{G} \to G$ such that the diagram

(1)



commutes. Then if $P \hookrightarrow G$ is Toeplitz, then the inclusion $P \hookrightarrow \tilde{G}$ must be Toeplitz as well. Proof: In our equivalent formulation of the Toeplitz condition, the only part which depends on the group embedding of our semigroup is the intersection $P \cap qp^{-1}P$. In our particular situation, the intersection $P \cap qp^{-1}P$ taken in $\tilde{G}$ is given by

$$\left\{x \in P \colon pq^{-1}x \in P \text{ in } \tilde{G}\right\},$$

while the intersection $P \cap qp^{-1}P$ taken in G is given by

$$\left\{x \in P \colon pq^{-1}x \in P \text{ in } G\right\}.$$

Because of the commutative diagram, the condition $pq^{-1}x \in P$ in $\tilde{G}$ implies the condition $pq^{-1}x \in P$ in G. Hence the intersection $P \cap qp^{-1}P$, taken in $\tilde{G}$, is contained in the intersection $P \cap qp^{-1}P$, taken in G, where we view both intersections as subsets of P. Our claim follows. □

Cor: Let $P \hookrightarrow G_{\text{univ}}$ be the universal group embedding. If $P \hookrightarrow G_{\text{univ}}$ is not Toeplitz, then there is no group embedding of P which is Toeplitz.

Counterex: Consider $P = \langle a, b \mid a = b^l a b^k \rangle$. $G_{\text{univ}} = \langle a, b \mid a = b^l a b^k \rangle$. If $k > 1$, then $P \subseteq G_{\text{univ}}$ is not Toeplitz: $g = aba^{-1}$. Since P right LCM (normal form), $gP \cap P = pP$ for some $p \in P$. $\#_a(p) = 0 \Rightarrow p = b^m$ for some $m \geq 0$. So $b^m = p \in gP \Rightarrow b^m = aba^{-1}x$ for some $x \in P$. Normal form $\Rightarrow x = ay$ for some $y \in P$. But then $b^m = aby$. Contradiction (compare $\#_a$). $ab^n \in gP \cap P$ for all $n \in \mathbb{Z}$ as $ab^{n-1} \in P$. So $a \in pP \Rightarrow \#_a(p) = 1$. So $p = b^i a b^j$, $j \in \mathbb{Z}$. Wlog $0 \leq i < l$. $\exists\, x_n \in P$: $px_n = ab^n$. Compare $\#_a \Rightarrow x_n = b^{k_n}$ for some $k_n \geq 0$. So $b^i a b^{j+k_n} = ab^n \Rightarrow i = 0$ & $j \leq j + k_n = n$. Contradiction ($n \in \mathbb{Z}$ arbitrary).

Definition: P satisfies the independence condition if for every $X, X_1, \ldots, X_n \in \mathcal{J}$, $X = \bigcup_{i=1}^n X_i$ implies that $X = X_i$ for some $1 \leq i \leq n$.

Ex: right LCM case: $\mathcal{J}^\times = \{pP\}$. It follows that P satisfies independence: Let P be a monoid ($e \in P$) and assume $pP = \bigcup_{i=1}^n p_i P$. Then $p = p \cdot e \in pP$, so $p \in p_i P$ for some $1 \leq i \leq n$. But $p_i P$ is a right ideal, so $p \in p_i P$ implies $pP \subseteq p_i P$. Hence $pP = p_i P$. This shows that P satisfies independence.

For a ring R of algebraic integers in a number field, $R \rtimes R^\times$ always satisfies independence (but it is not necessarily right LCM). The proof uses facts about the ideal structure of R, that R is a Dedekind domain.

Counterex: $P = \mathbb{N} \setminus \{1\}$ does not satisfy independence: We have the following constructible right ideals

$$2 + P = \{2, 4, 5, 6, \ldots\} \text{ and } 3 + P = \{3, 5, 6, 7, \ldots\}.$$

Hence

$$5 + \mathbb{N} = \{5, 6, 7, \ldots\} = (2 + P) \cap (3 + P)$$

is also a constructible right ideal of P. Moreover, it is clear that

$$5 + \mathbb{N} = (5 + P) \cup (6 + P).$$

But since $5 + P \subsetneq 5 + \mathbb{N}$ and $6 + P \subsetneq 5 + \mathbb{N}$, it follows that P does not satisfy independence.

A similar argument shows that for every numerical semigroup of the form $\mathbb{N} \setminus F$, where F is a non-empty finite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus F$ is still closed under addition, the independence condition does not hold.

However, this can fail for rings which are not integrally closed. For instance, $R = \mathbb{Z}[i\sqrt{3}]$ is not integrally closed in $\mathbb{Q}[i\sqrt{3}]$. Its integral closure is $\mathbb{Z}[\frac{1}{2}(1 + i\sqrt{3})]$. And indeed, $R \rtimes R^\times$ does not satisfy independence.

Lemma: If $P \subseteq G$ is Toeplitz, then

(i) for all g in G and X in $\mathcal{J}$, $P \cap (g \cdot X)$ lies in $\mathcal{J}$,
(ii) $\bar{\mathcal{J}} = \{g \cdot X \colon g \in G, X \in \mathcal{J}\}$ (i.e. intersections are not needed).

If in addition $\mathcal{J}$ is independent, then

(iii) $\bar{\mathcal{J}}$ is independent.

(i) $P \cap (gX) = l_g(X) = l_{q_1}^{-1} l_{p_1} \cdots l_{q_N}^{-1} l_{p_N}(X) =$— $X = \mathrm{ran}(\varphi)$ — $= \mathrm{ran}(l_{q_1}^{-1} l_{p_1} \cdots l_{q_N}^{-1} l_{p_N} \varphi)$.

(ii): we just have to show that the right hand side in (ii) is closed under finite intersections. Take g_1, g_2 in G and X_1, X_2 in $\mathcal{J}$. Then $(g_1 \cdot X_1) \cap (g_2 \cdot X_2) = g_1 \cdot (X_1 \cap ((g_1^{-1} g_2) \cdot X_2)) = g_1 \cdot (X_1 \cap \underbrace{P \cap (g_1^{-1} g_2) \cdot X_2}_{\in \mathcal{J} \text{ by (i)}})$ is of the desired form by (i).

(iii): By (ii), it suffices to prove that given g, $g_1, \dots, g_n$ in G and X, $X_1, \dots X_n$ in $\mathcal{J}$ such that $g \cdot X = \bigcup_{i=1}^n g_i \cdot X_i$, we must have $g \cdot X = g_i \cdot X_i$ for some i. Now $g \cdot X = \bigcup_{i=1}^n g_i \cdot X_i$ implies $X = \bigcup_{i=1}^n (g^{-1} g_i) \cdot X_i$. In particular, since $X \subseteq P$, we must have $(g^{-1} g_i) \cdot X_i \subseteq P$ for all $1 \leq i \leq n$. Therefore $(g^{-1} g_i) \cdot X_i = P \cap ((g^{-1} g_i) \cdot X_i)$ lies in $\mathcal{J}$ by (i). As $\mathcal{J}$ is independent, there exists i such that $X = (g^{-1} g_i) \cdot X_i$. Thus $g \cdot X = g_i \cdot X_i$. □

$\bar{D} :=$ smallest G-invariant subalgebra of $\ell^\infty(G)$ containing 1_P. $\bar{\mathcal{J}} := \{\bigcup g_i P\} \cup \{\emptyset\}$. $D = \overline{\mathrm{span}}(1_{\mathcal{J}}) \Rightarrow \bar{D} = \overline{\mathrm{span}}(1_{\bar{\mathcal{J}}})$. $\bar{D} \rtimes_r G = \overline{\mathrm{span}}\,\{d\lambda_g\}$. Claim: $P \subseteq G$ Toeplitz $\Rightarrow C_\lambda^*(P) = 1_P(\bar{D} \rtimes_r G)1_P$. "$\subseteq$": $V_p = 1_P \lambda_p 1_P$. "$\supseteq$": suffices to show $1_P 1_Y \lambda_g 1_P \in C_\lambda^*(P)$ for $Y = \bigcup_i g_i X_i$. $1_P 1_Y \lambda_g 1_P = 1_P 1_Y 1_P (1_P \lambda_g 1_P) = 1_P (\prod_i \lambda_{g_i} 1_{X_i} \lambda_{g_i}^*) 1_P (1_P \lambda_g 1_P) = \prod_i ((1_P \lambda_{g_i} 1_P) 1_{X_i} (1_P \lambda_{g_i} 1_P)^*)(1_P \lambda_g 1_P)$. So it suffices to show $1_P \lambda_g 1_P \in C_\lambda^*(P)$. But $1_P \lambda_g 1_P \,\hat{=}\, g^{-1} P \cap P \cong P \cap gP$, $x \mapsto gx$ RHS $\in I_l$ by Toeplitz $\Rightarrow$ LHS $= V_{q_1}^* V_{p_1} \cdots V_{q_N}^* V_{p_N} \in C_\lambda^*(P)$. QED. **Cor**: $P \subseteq G$ Toeplitz $\Rightarrow C_\lambda^*(P) \sim_M \bar{D} \rtimes_r G$.

$\bar{\Omega} := \mathrm{Spec}\, \bar{D}$. So: If $P \subseteq G$ is Toeplitz and P sat. independence, then $G \curvearrowright \bar{\Omega}$ sat. independence (take $\mathcal{V}$:= compact open sets in $\bar{\Omega}$ given by supports of $1_Y \in \bar{D} = C_0(\bar{\Omega})$, $Y \in \bar{\mathcal{J}}$) and $C_\lambda^*(P) \sim_M C_0(\bar{\Omega}) \rtimes_r G \Rightarrow K_*(C_\lambda^*(P)) \cong K_*(C_0(\bar{\Omega}) \rtimes_r G)$. So we can apply our general K_* formula.

Theorem (Cuntz-Echterhoff-L): $P \subseteq G$ Toeplitz, P sat. independence, G sat. BCwC. Then

$$K_*(C_\lambda^*(P)) \cong \bigoplus_{[Y] \in G \backslash \bar{\mathcal{J}}^\times} K_*(C_\lambda^*(G_Y)), \text{ where } G_Y = \{g \in G \colon gY = Y\}.$$

Examples: Let P be a right LCM monoid ($\mathcal{J}^\times = \{pP \colon p \in P\}$), which imbeds into a group G such that $P \subseteq G$ is Toeplitz. Then $\mathcal{J}_{P \subseteq G}^\times = \{gP \colon g \in G\}$. Hence $G \backslash \mathcal{J}_{P \subseteq G}^\times = \{[P]\}$. Moreover, $G_P = P^*$. So if G satisfies the Baum-Connes conjecture with coefficients, then $K_*(C_\lambda^*(P)) \cong K_*(C_\lambda^*(P^*))$.

Let $P = R \rtimes R^\times$, where R is the ring of algebraic integers in a number field K. P embeds into $G = K \rtimes K^\times$. G is amenable, hence satisfies the Baum-Connes conjecture with coefficients. Moreover, we can canonically identify $G \backslash \mathcal{J}_{P \subseteq G}^\times$ with Cl_K, the class group of K. This is a very important invariant in number theory, which people would like to understand better. This is a group of equivalence classes of non-zero ideals of R. Given such an ideal $\mathfrak{a}$, $[\mathfrak{a}]$ corresponds to $\mathfrak{a} \times \mathfrak{a}^\times \subseteq R \rtimes R^\times$, and $G_{\mathfrak{a} \times \mathfrak{a}^\times} = \mathfrak{a} \rtimes R^*$. So

$$K_*(C_\lambda^*(R \rtimes R^\times)) \cong \bigoplus_{[\mathfrak{a}] \in Cl_K} K_*(C_\lambda^*(\mathfrak{a} \rtimes R^*)).$$

Applications: We start with a classification result for semigroup C*-algebras of RAAM. We can always decompose Γ into co-irreducible components, i.e., co-irreducible subgraphs $\Gamma_i = (V_i, E_i)$. Let $t(\Gamma)$ be the number of those co-irreducible components which are singletons. Moreover, let us define the Euler characteristic (or rather $1 - \chi$) of a finite component $\Gamma_i = (V_i, E_i)$. To this end, we view Γ_i as a simplicial complex by defining for every $n = 0, 1, 2, \dots$ the set K_n of n-simplices by all complete subgraphs of Γ_i with $n+1$ vertices. Then we set $\chi(\Gamma_i) := 1 - \sum_{n=0}^{\infty} (-1)^n |K_n|$. Given a graph Γ and $k \in \mathbb{Z}$, let $N_k(\Gamma)$ be the number of co-irreducible components of Γ with Euler characteristic equal to k.

Theorem (Eilers-L-Ruiz): Let Γ and Λ be finite graphs. The following are equivalent:

(1) $C^*_\lambda(A^+_\Gamma) \cong C^*_\lambda(A^+_\Lambda)$
(2) (a) $t(\Gamma) = t(\Lambda)$
  (b) $N_k(\Gamma) + N_{-k}(\Gamma) = N_k(\Lambda) + N_{-k}(\Lambda)$ for all $k \in \mathbb{Z}$
  (c) $N_0(\Gamma) > 0$ or

$$\sum_{k>0} N_k(\Gamma) \equiv \sum_{k>0} N_k(\Lambda) \mod 2.$$

We remark that it is not necessary to restrict to finite graphs.

Why Euler characteristic? Γ finite $\Rightarrow A^+_\Gamma$ finitely generated, so: $0 \to \mathcal{K}(\ell^2 P) \to C^*_\lambda(A^+_\Gamma) \to Q \to 0$. In K-theory, get $0 \to K_1(Q) \to \mathbb{Z} \to \mathbb{Z} \to K_0(Q) \to 0$. Generator of first $\mathbb{Z}$: $1_e = 1_P - 1_{\bigcup_{v \in V} \sigma_v P} = 1_P - \sum_{\emptyset \neq U \subseteq V} (-1)^{|U|} 1_{\bigcap_{v \in U} \sigma_v P}$. Hence in K_0, the map $\mathbb{Z} \to \mathbb{Z}$ is given by mult. with $\chi(\Gamma)$. Hence $K_0(Q) \cong \mathbb{Z}/|\chi(\gamma)|\mathbb{Z}$.

Here is another classification result, this time for semigroup C*-algebras attached to $ax+b$-semigroups over rings of algebraic integers in number fields.

Theorem (L, Bruce-L): Let K and L be number fields with rings of algebraic integers R and S. If $C^*_\lambda(R \rtimes R^\times) \cong C^*_\lambda(S \rtimes S^\times)$ then $\zeta_K = \zeta_L$.

Assume, in addition, that K and L are Galois extensions. In that case, we have $C^*_\lambda(R \rtimes R^\times) \cong C^*_\lambda(S \rtimes S^\times)$ if and only if $K \cong L$.