A Twisted Local Index Formula for Curved Noncommutative Two Tori

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Outline

1. Motivation
   - Background
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2. Our Results/Contribution
   - Extension to f.g. proj. right modules (vector bundles)
   - Basic Ideas for Calculation

The Noncommutative 2-Torus

- Fix $\theta \in \mathbb{R} \backslash \mathbb{Q}$. The *NC 2-torus algebra* A_θ is the universal C^*-algebra generated by two unitary elements U and V satisfying $VU = UV \exp(2\pi i\theta)$.
- Define the trace $\tau : A_\theta \to \mathbb{C}$ by $\tau(U^n V^m) = 0$ for $(n, m) \neq 0$ and $\tau(1) = 1$.
- Define an inner product $\langle \cdot, \cdot \rangle : A_\theta \times A_\theta \to \mathbb{C}$ by $\langle a, b \rangle = \tau(b^* a)$ with induced norm $|| \cdot || : A_\theta \to \mathbb{R}_{\geq 0}$.
- Define derivations δ_1 and δ_2 by the relations $\delta_1(U) = U$, $\delta_2(V) = V$, and $\delta_1(V) = \delta_2(U) = 0$. Let $\partial = \delta_1 + i\delta_2$, $\partial^* = \delta_1 - i\delta_2$, and $\triangle = \partial^*\partial = \delta_1^2 + \delta_2^2$.

ΨD Operators on the NC 2-Torus

- Let $\{\alpha_s\}_{s\in\mathbb{R}^2}$ be a 2-parameter group of automorphisms given by $U^nV^m \mapsto e^{i(s_1n+s_2m)}U^nV^m$ (analog of inv. Fourier).
- We define the subalgebra A_θ^∞ of smooth elements of A_θ^∞ to be those $a \in A_\theta$ such that $s \mapsto \alpha_s(a)$ is smooth or, equivalently, those $a \in A_\theta$ expressible by an expansion of the form $\sum_{(n,m)\in\mathbb{Z}^2} a_{n,m}U^nV^m$, where the sequence $\{a_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$ is in the Schwartz space $\mathcal{S}(\mathbb{Z}^2)$, i.e. for all $(p,q)\in\mathbb{Z}^2$, $\sup_{(n,m)\in\mathbb{Z}^2}(|n|^p|m|^q|a_{n,m}|) < \infty$.
- A symbol $\rho \in C^\infty(\mathbb{R}^2, A_\theta^\infty)$ has a corresponding ΨD *operator* P_ρ defined by the *oscillatory integral*

$$a \mapsto \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-is\cdot\xi}\rho(\xi)\alpha_s(a)\,\mathrm{d}s\,\mathrm{d}\xi.$$

Definition of Oscillatory Integral [Ray91]

Definition

Let q be a nondegenerate real quadratic form on $\mathbb{R}^n$, a be a C^∞ complex-valued function defined on $\mathbb{R}^n$ such that the functions $(1+|x|^2)^{-m/2}\partial^\alpha a(x)$ are bounded on $\mathbb{R}^n$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$, and φ be a Schwartz function, i.e. the functions $x^\alpha \partial^\beta \varphi(x)$ are bounded on $\mathbb{R}^n$ for all pairs $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. Suppose further that $\varphi(0) = 1$. Then the limit $\lim_{\epsilon\to 0} \int e^{iq(x)} a(x)\varphi(\epsilon x)\,\mathrm{d}x$ exists, is independent of φ (as long as $\varphi(0) = 1$), and is equal to $\int e^{iq(x)} a(x)\,\mathrm{d}x$ when $a \in L^1$. When $a \notin L^1$, we continue to denote this limit by $\int e^{iq(x)} a(x)\,\mathrm{d}x$, and have an estimate $\left|\int e^{iq(x)} a(x)\,\mathrm{d}x\right| \leq C_{q,m} \max_{|\alpha|\leq m+n+1} \inf\{U \in \mathbb{R} : |(1+|x|^2)^{-m/2}\partial^\alpha a| \leq U \text{ almost everywhere}\}$ where $C_{q,m}$ depends only on the quadratic form q and the order m.

ΨD Calculus on the NC 2-Torus [Tao19]

Suppose P is a ΨDO on A_θ^∞ with symbol $\sigma(P) = \rho = \rho(\xi)$ of order M_1, i.e. $||\rho(\xi)||_0 \leq C_\rho(1 + |\xi|)^{M_1}$ for some $C_\rho > 0$, and Q is a ΨDO on A_θ^∞ with symbol $\sigma(Q) = \phi = \phi(\xi)$ of order M_2.

- The symbol of the adjoint P^* is of order M_1 and satisfies

$$\sigma(P^*) \sim \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2_{\geq 0}} \frac{\partial_1^{\ell_1}\partial_2^{\ell_2}\delta_1^{\ell_1}\delta_2^{\ell_2}[(\rho(\xi))^*]}{\ell_1!\ell_2!}.$$

- The symbol of the product QP is of order $M_1 + M_2$ and satisfies

$$\sigma(QP) \sim \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2_{\geq 0}} \frac{\partial_1^{\ell_1}\partial_2^{\ell_2}\phi(\xi)\delta_1^{\ell_1}\delta_2^{\ell_2}\rho(\xi)}{\ell_1!\ell_2!}.$$

The Gauss–Bonnet Theorem on the NC 2-Torus

- The *zeta function* of an operator $\mathcal{O}$ is the function

$$\zeta_{\mathcal{O}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty \mathrm{Trace}^+(e^{-t\mathcal{O}}) t^{s-1}\, \mathrm{d}t$$

  defined by Mellin transform for $\mathrm{Re}(s) > 1$, where $\mathrm{Trace}^+(e^{-t\mathcal{O}}) := \mathrm{Trace}(e^{-t\mathcal{O}}) - \dim\ker\mathcal{O}$, and extended by meromorphic continuation to all s, with a simple pole at 1.
- [CT11] Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$ and $k = \exp(h/2)$ where $h = h^* \in A_\theta^\infty$. Then the value at the origin of the zeta function $\zeta(s)$ of the Laplacian $\triangle' \sim k\triangle k$ is independent of k.
- [FK12] $\zeta(0) = -1$.

Algorithm for Calculation in Proof

- By complex analysis, $\zeta(0) + \dim\ker \triangle' = \int_{\mathbb{R}^2} \tau(b_2(\xi))\, \mathrm{d}\xi$ where B_λ is a ΨDO approximating $(\lambda \mathbf{1} - \triangle')^{-1}$ with symbol expansion $\sigma(B_\lambda) = b_0(\xi) + b_1(\xi) + b_2(\xi)$.
- This symbol expansion maybe calculated recursively starting with $b_0(\xi) = (\lambda - k^2|\xi|^2)^{-1}$.
- Using the ΨD calculus, we finds that $b_2 b_0^{-1}$ is a sum of terms with b_0 all on the left, b_0 in the middle, and b_0^2 in the middle. The trace property and integration by parts allow us to use a rearrangement lemma, proven by Fourier analysis in [CT11], that gives us a closed formula for

$$\int_0^\infty (k^2 u + 1)^{m+1} \backslash (k^{2m+2} u^m \rho)/(k^2 u + 1)\, \mathrm{d}u.$$

Constructing a twisted spectral triple

We follow Ponge's construction in [PW16] but may restrict to cyclic modules (line bundles) for $A := A_\theta$ [PV80, Rie83]:

- Form a Hilbert space $\mathcal{H}$ from A by completing w.r.t. $\langle\cdot,\cdot\rangle$.
- Let $e^2 = e \in A$ so that $e\mathcal{H}$ is a Hilbert space with the induced inner product. We get an orthogonal splitting $e\mathcal{H} = e\mathcal{H}^+ \oplus e\mathcal{H}^-$ where
  - $\mathcal{H}^+ := \mathcal{H}_\varphi$ is the completion of A with respect to the inner product $\langle\cdot,\cdot\rangle_\varphi$ defined by the functional

    $$\varphi(a) := \tau(a\exp(-h))$$

    where $h = h^* \in A^\infty$
  - $\mathcal{H}^- := \mathcal{H}^{(1,0)}$ is the completion of the $\mathbb{C}$-vector space of finite sums of the form $\sum a\partial(b)$ w.r.t. the inner product

    $$\langle a\partial b, c\partial d\rangle := \tau(c^* a(\partial b)(\partial d)^*)$$

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Twisting the spectral triple

- The Dirac operator on $\mathcal{H}$ is

$$D := \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix}$$

  where $\partial_\varphi : \mathcal{H}^+ \to \mathcal{H}^-$ is a restriction of ∂ and $\partial_\varphi^* = R_{k^2}\partial^*$.
- Let $\sigma(a) := k^{-1}ak$ and let $a \in A^{\mathrm{op}}$ act on $\mathcal{H}$ via $\pi : A^{\mathrm{op}} \to \mathrm{End}(\mathcal{H})$ where $\pi(a)(b_1, b_2) = (b_1\sigma(a), b_2 a)$.
- $(A, \mathcal{H}, D)_\sigma$ as constructed forms a twisted spectral triple since the twisted commutator $Da^{\mathrm{op}} - \sigma(a^{\mathrm{op}})D$ is a bounded operator on $\mathcal{H}$.

Dirac operator twisted by a module (vector bundle)

- Consider the operator $D_{e,\sigma} = \sigma(e)De : e\mathcal{H} \to \sigma(e)\mathcal{H}$, which is D twisted with a f.g. proj. module (vector bundle) $e\mathcal{H}$ by means of the connection σ.
- For any $a \in e\mathcal{H}$,

$$D_{e,\sigma}a = \sigma(e)De \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \partial_\varphi^*(a_2 e)k^{-1}\sigma(e)k \\ \partial_\varphi(a_1\sigma(e))\sigma(e) \end{pmatrix},$$

so

$$D_{e,\sigma} := \begin{pmatrix} 0 & D_{e,\sigma}^- \\ D_{e,\sigma}^+ & 0 \end{pmatrix}$$

where

$$D_{e,\sigma}^+ = \sigma(e)\partial_\varphi e : e\mathcal{H}^+ \to \sigma(e)\mathcal{H}^-$$

and

$$D_{e,\sigma}^- = \sigma(e)\partial_\varphi^* e : e\mathcal{H}^- \to \sigma(e)\mathcal{H}^+.$$

Our Goal

- We wish to calculate $\mathrm{ind}(D^+_{e,\sigma}) := \dim\ker(D^+_{e,\sigma})^* D^+_{e,\sigma} - \dim\ker D^+_{e,\sigma}(D^+_{e,\sigma})^*$.
- $(D^+_{e,\sigma})^* D^+_{e,\sigma}$ has the same nullity as $L^+_1 = e\partial^*_\varphi \sigma(e)\partial_\varphi : \mathcal{H}^+ \to \mathcal{H}^+$ which is anti-unitarily equivalent to

$$L^+ = k^{-1}\sigma(e)k^2\partial\sigma(e)\partial^* k : \mathcal{H}_0 \to \mathcal{H}_0$$

- $D^+_{e,\sigma}(D^+_{e,\sigma})^*$ has the same nullity as $L^-_1 = \sigma(e)\partial_\varphi e\partial^*_\varphi : \mathcal{H}^- \to \mathcal{H}^-$ which is anti-unitarily equivalent to

$$L^- = \sigma(e)\partial^*\sigma(e)k^2\partial : \mathcal{H}_0 \to \mathcal{H}_0.$$

Strategy for the noncommuting idempotent [Tao19]

- It's essentially the same algorithm as before, but instead of $b_0(\xi) = (\lambda - k^2|\xi|^2)^{-1}$ for $\triangle'$ we have $b_0^+(\xi) = (\lambda - \sigma^2(ek^2e)|\xi|^2)^{-1}$ for L^+ and $b_0^-(\xi) = (\lambda - \sigma(ek^2)|\xi|^2)^{-1}$ for L^-.
- It helps to use the concept of a Hadamard product of generating functions from analytic combinatorics [FS09]:

$$(1 + ek^2u)^{-j} = (1 + k^2u)^{-j} \odot \frac{1}{1 - u\Delta R_e}(1),$$

where $\sum f_n z^n \odot \sum g_n z^n := \sum f_n g_n z^n$ and $\Delta(a) = k^{-2}ak^2$.

Hadamard Product Integral Parameterization

- As a Hadamard product $(ek^2u+1)^{-j}$

$$
\begin{aligned}
&= \frac{1}{(1+k^2u)^j} \odot \frac{1}{1-u\Delta R_e}(1) \\
&= \frac{1}{2\pi i}\int_C \frac{1}{(1+k^2u/z)^j}\,\frac{1}{1-z\Delta R_e}(1)\frac{\mathrm{d}z}{z} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{[1+\exp(h+s-a-i\theta)]^j}\,\frac{1}{1-\Delta R_e\exp(a+i\theta)}\,\mathrm{d}\theta,
\end{aligned}
$$

  where C is a circle of radius $\exp(a) > 1$ centered at 0.
- The proof of the rearrangement lemma for L^+ reduces to that of L^-, which reduces to that of $\triangle'$ in [CT11].

Reduction to $k = 1$ and the Connes–Chern number

- For $k = 1$ and other values of k that leave $D^+_{e,\sigma}$ in the same connected component of Fredholm operators, we get

$$\mathrm{Ind}\, D^+_{e,\sigma} = 2\pi i\, \tau\left(e\delta_1(e)\delta_2(e) - e\delta_2(e)\delta_1(e)\right).$$

- Note that

$$c_1(e) := 2\pi i\, \tau(e\delta_1(e)\delta_2(e) - e\delta_2(e)\delta_1(e))$$

  is the Connes–Chern number of our projection e.

- As proven in [DJLL18], one can construct e as a self-dual or anti-self dual projection with $c_1(e) = n$ for any $n \in \mathbb{Z}$.

Summary

- We extended Connes and Tretkoff's rearrangement lemma to cases with a noncommuting idempotent using Hadamard products from analytic combinatorics.
- We showed that the index of $D^+_{e,\sigma}$ is the Connes-Chern number of our projection e for $k = 1$ and other values of k that leave $D^+_{e,\sigma}$ in the same connected component of Fredholm operators.
- Outlook/What remains to be done:
  - Simplify the local formula as was done in [FK13] for $e = 1$.
  - Relate functions of Δ under stability of the index, cf. [CF16].
  - Open Problem: Show how the Hadamard product interacts with the divided differences mentioned in [Les17].

For Further Reading I

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