C*-algebras coming from cube complexes and buildings

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Oslo
November 2019

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Arithmetic side of buildings

Higher dimensional words

Higher-dimensional Ramanujan cube complexes

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nD polyhedral C*-algebras

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Buildings and polyhedra

Buildings consist of **chambers** and **apartments**.

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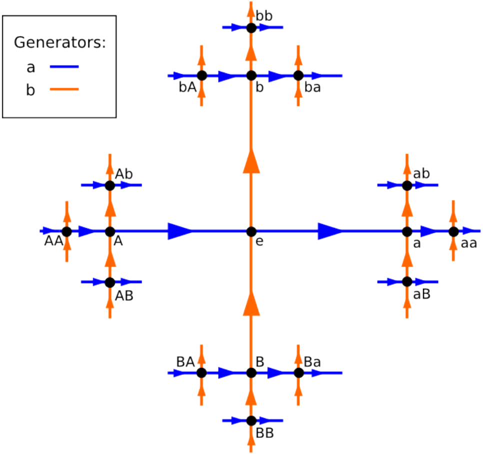
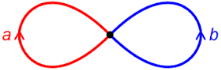
- X is a union of tessellated nD-spaces (apartments)
- for any two chambers there is an apartment containing both of them
- if two apartments F_1 and F_2 have non-trivial intersection, then there is an isomorphism from F_1 to F_2, fixing $F_1 \cap F_2$ pointwise.

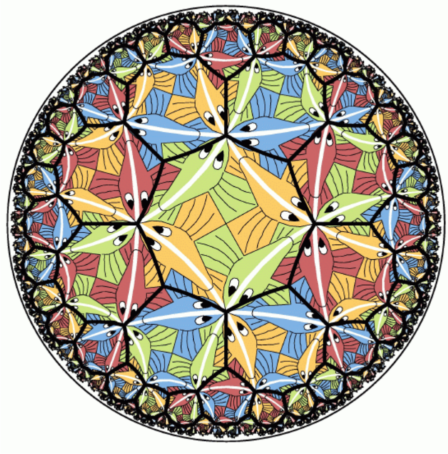
One-dimensional buildings: Cayley graphs of free groups





The four-valent tree is the *universal cover* of the wedge of two circles.

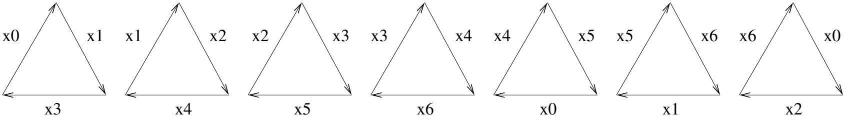
Example of an apartment: M.C.Escher - Circle Limit III



Polyhedra and links

Definition

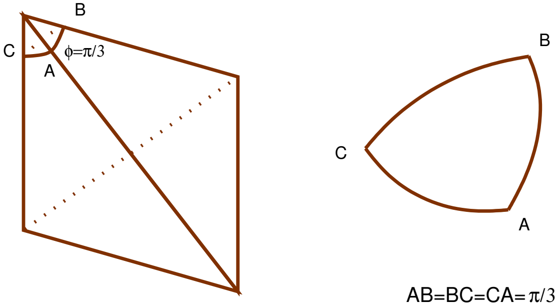
A *polyhedron* is a two-dimensional complex which is obtained from several decorated p-gons by identification of corresponding sides.



Polyhedra and links

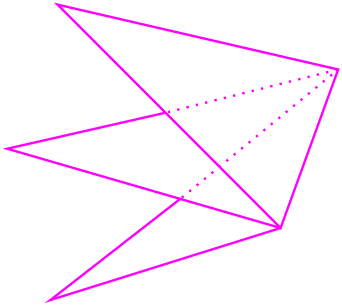
Definition

Take a sphere of a small radius at a point of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

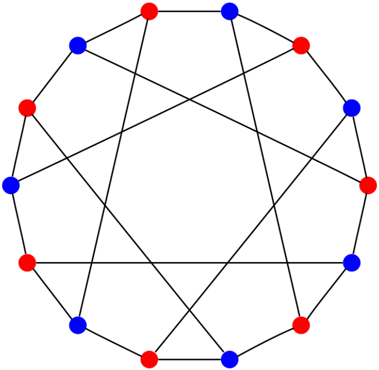


Polyhedra and links

We consider *thick* polyhedra, which means that each edge is contained in at least three polygons.



Example of a link



This graph has *diameter* (the maximal distance between two vertices) three and *girth* (the length of the shortest cycle) six.

Polyhedra and links

Theorem (Ballmann, Brin 1994)

Let X be a compact two-dimensional thick polyhedron. If all links are graphs of diameter m and girth $2m$, then the universal cover of the polyhedron is a two-dimensional building.

A polygonal presentation is a set of words satisfying certain combinatorial properties (AV, 2000).

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Theorem (AV,2002)

A polyhedron with given links can be constructed explicitly using a polygonal presentation. Any connected bipartite graph can be realized as a link of every vertex a 2-dimensional polyhedron with $2k$-gonal faces.

A Result of Jacobi

In p is an odd prime, the number of

$$(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$$

such that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$$

is

$$8(p+1).$$

Suppose $p \equiv 1 \pmod 4$. Then exactly one a_j is odd and the number of representations with a_0 odd, $a_0 > 0$, is

$$p+1.$$

Consequence:
Let S_p be the set of integer quaternions

$$a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}(\mathbb{Z})$$

with $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$ and $a_0 > 0$, a_0 odd, $|a|^2 = p$. Then

$$|S_p| = p+1.$$

An Arithmetic Construction

If $p \equiv 1(\text{mod } 4)$ is prime, then $x^2 \equiv -1(\text{mod p})$ has a solution in $\mathbb{Z}$, so, by Hensel's Lemma, $x^2 = -1$ has a solution i_p in $\mathbb{Q}_p$.

Define

$$\psi_p : \mathbb{H}(\mathbb{Z}) \mapsto PGL_2(\mathbb{Q}_p)$$

by

$$\psi_p(a) = \begin{pmatrix} a_0 + a_1 i_p & a_2 + a_3 i_p \\ -a_2 + a_3 i_p & a_0 - a_1 i_p \end{pmatrix}$$

Theorem (Lubotzky, Phillips, Sarnak; Margulis1988)

$\psi_p(S_p)$ contains $p+1$ elements and generates a free group Γ_p of rank $(p+1)/2$. Γ_p acts freely and transitively on the vertices of the $(p+1)$-regular tree T_{p+1}. Ramanujan graphs are Cayley graphs of $PGL_2(\mathbb{Z}/q\mathbb{Z})$ with respect to generators $\psi_p(S_p)$ for $p \neq q$.

Arithmetic constructions on products of trees

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- Infinite series of lattices acting on products of trees of different valencies: Burger-Mozes (2000), Rattaggi (2004)
- Arithmetic lattices acting on products of trees of the same valency: joint work with Jakob Stix (2013) (and later developments)
- The same valency is needed to get Ramanujan complexes.

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Let $\mathcal{B}$ be a n-dimensional Euclidean building equipped with a cocompact action of a group G. nD-dimensional words are rectangular subsets of apartments in $\mathcal{B}$, decorated by the action of G.

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Definition

The boundary Ω of $\mathcal{B}$ is isomorphic to equivalence classes of sectors in $\mathcal{B}$

Cubes and Products of Trees

The four squares define a group G which belongs to a family constructed by J.Stix and AV

$$G = \langle a_1, a_2, b_1, b_2 \mid a_2 b_1 a_2 b_2^{-1}, a_1 b_2^{-1} a_2^{-1} b_2^{-1}, a_1 b_1 a_1 b_2, a_1 b_1^{-1} a_2 b_1^{-1} \rangle.$$

Let $S = \{a_1, a_2, b_1, b_2\}$. Then $Cay(G, S)$ is a one-skeleton of a thick Euclidean building (product of two trees) with the following properties:

- lots of (equilateral) squares (chambers)

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- G is an arithmetic lattice in $PGL(2, \mathbb{F}_3((t))) \times PGL(2, \mathbb{F}_3((t)))$

3D example

Hurwitz quaternions can be used to get a cube complex of any dimension, for any set of odd primes (RSV 2018).

$a_1 = 1 + j + k$, $a_2 = 1 + j - k$, $a_3 = 1 - j - k$, $a_4 = 1 - j + k$,
$b_1 = 1 + 2i$, $b_2 = 1 + 2j$, $b_3 = 1 + 2k$, $b_4 = 1 - 2i$, $b_5 = 1 - 2j$, $b_6 = 1 - 2k$,
$c_1 = 1 + 2i + j + k$, $c_2 = 1 - 2i + j + k$, $c_3 = 1 + 2i - j + k$, $c_4 = 1 + 2i + j - k$,
$c_5 = 1 - 2i - j - k$, $c_6 = 1 + 2i - j - k$, $c_7 = 1 - 2i + j - k$, $c_8 = 1 - 2i - j + k$.

With this notation we have $a_i^{-1} = a_{i+2}$, $b_i^{-1} = b_{i+3}$, and $c_i^{-1} = c_{i+4}$, and using these abbreviations we find the explicit presentation.

3D example

$$\Gamma_{\{3,5,7\}} = \left\langle \begin{array}{c} a_1, a_2 \\ b_1, b_2, b_3 \\ c_1, c_2, c_3, c_4 \end{array} \;\middle|\; \begin{array}{c} a_1b_1a_4b_2,\ a_1b_2a_4b_4,\ a_1b_3a_2b_1, \\ a_1b_4a_2b_3,\ a_1b_5a_1b_6,\ a_2b_2a_2b_6 \\ a_1c_1a_2c_8,\ a_1c_2a_4c_4,\ a_1c_3a_2c_2,\ a_1c_4a_3c_3, \\ a_1c_5a_1c_6,\ a_1c_7a_4c_1,\ a_2c_1a_4c_6,\ a_2c_4a_2c_7 \\ b_1c_1b_5c_4,\ b_1c_2b_1c_5,\ b_1c_3b_6c_1, \\ b_1c_4b_3c_6,\ b_1c_6b_2c_3,\ b_1c_7b_1c_8, \\ b_2c_1b_3c_2,\ b_2c_2b_5c_5,\ b_2c_4b_5c_3, \\ b_2c_7b_6c_4,\ b_3c_1b_6c_6,\ b_3c_4b_6c_3 \end{array} \right\rangle.$$

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- From classical Hamiltonian quaternions, we've moved to more general quaternionic algebras, namely, the quaternionic multiplication is the following:

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Example

For $p = 3$ we get four squares with labels

$$a_2 b_1 a_2 b_2^{-1}, a_1 b_2^{-1} a_2^{-1} b_2^{-1}, a_1 b_1 a_1 b_2, a_1 b_1^{-1} a_2 b_1^{-1}, where$$

$$a_1 = t + \mathbf{j} + \mathbf{k}, a_2 = t + \mathbf{j} - \mathbf{k}, b_1 = t + \mathbf{j}, b_2 = t + \mathbf{k}.$$

Arithmetic lattices acting simply transitively on products of trees

Let q be a prime power. Let

$$\delta \in \mathbb{F}_{q^2}^{\times}$$

be a generator of the multiplicative group of the field with q^2 elements. If $i, j \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ are

$$i \not\equiv j \pmod{q-1},$$

then $1 + \delta^{j-i} \neq 0$, since otherwise

$$1 = (-1)^{q+1} = \delta^{(j-i)(q+1)} \neq 1,$$

a contradiction. Then there is a unique $x_{i,j} \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ with

$$\delta^{x_{i,j}} = 1 + \delta^{j-i}.$$

With these $x_{i,j}$ we set $y_{i,j} := x_{i,j} + i - j$, so that

$$\delta^{y_{i,j}} = \delta^{x_{i,j}+i-j} = (1 + \delta^{j-i}) \cdot \delta^{i-j} = 1 + \delta^{i-j}.$$

We set

$$l(i,j) := i - x_{i,j}(q-1),$$
$$k(i,j) := j - y_{i,j}(q-1).$$

Let $M \subseteq \mathbb{Z}/(q^2-1)\mathbb{Z}$ be a union of cosets stable under multiplication by q, and by addition of $q-1$.

Theorem (RSV 2018)

Each group $\Gamma_{M,\delta}$ acts simply transitively on a product of $d = |M|$ trees.

$$\Gamma_{M,\delta} = \left\langle a_i \textit{ for all } i \in M \;\middle|\; \begin{array}{c} a_{i+(q^2-1)/2}a_i = 1 \textit{ for all } i \in M, \\ a_ia_j = a_{k(i,j)}a_{l(i,j)} \textit{ for all } i,j \in M \textit{ with } i \not\equiv j \pmod{q-1} \end{array} \right\rangle$$

if q is odd, and if q is even:

$$\Gamma_{M,\delta} = \left\langle a_i \textit{ for all } i \in M \;\middle|\; \begin{array}{c} a_i^2 = 1 \textit{ for all } i \in M, \\ a_ia_j = a_{k(i,j)}a_{l(i,j)} \textit{ for all } i,j \in M \textit{ with } i \not\equiv j \pmod{q-1} \end{array} \right\rangle.$$

3D example

$$\Gamma = \left\langle \begin{array}{c} a_1, a_5, a_9, a_{13}, a_{17}, a_{21}, \\ b_2, b_6, b_{10}, b_{14}, b_{18}, b_{22}, \\ c_3, c_7, c_{11}, c_{15}, c_{19}, c_{23} \end{array} \middle| \begin{array}{c} a_i a_{i+12} = b_i b_{i+12} = c_i c_{i+12} = 1 \text{ for all } i, \\ a_1 b_2 a_{17} b_{22},\ a_1 b_6 a_9 b_{10},\ a_1 b_{10} a_9 b_6, \\ a_1 b_{14} a_{21} b_{14},\ a_1 b_{18} a_5 b_{18},\ a_1 b_{22} a_{17} b_2, \\ a_5 b_2 a_{21} b_6,\ a_5 b_6 a_{21} b_2,\ a_5 b_{22} a_9 b_{22}, \\ a_1 c_3 a_{17} c_3,\ a_1 c_7 a_{13} c_{19},\ a_1 c_{11} a_9 c_{11}, \\ a_1 c_{15} a_1 c_{23},\ a_5 c_3 a_5 c_{19},\ a_5 c_7 a_{21} c_7, \\ a_5 c_{11} a_{17} c_{23},\ a_9 c_3 a_{21} c_{15},\ a_9 c_7 a_9 c_{23}, \\ b_2 c_3 b_{18} c_{23},\ b_2 c_7 b_{10} c_{11},\ b_2 c_{11} b_{10} c_7, \\ b_2 c_{15} b_{22} c_{15},\ b_2 c_{19} b_6 c_{19},\ b_2 c_{23} b_{18} c_3, \\ b_6 c_3 b_{22} c_7,\ b_6 c_7 b_{22} c_3,\ b_6 c_{23} b_{10} c_{23}. \end{array} \right\rangle.$$

Adjacency operators for graphs and Ramanujan graphs

Let X be a connected graph with uniformly bounded valencies. We consider X as a 1-dimensional cubical complex and write X_0 for the set of vertices of X. We write $P \sim Q$ if two vertices $P, Q \in V(X)$ are adjacent, and we denote by $\mu(P, Q)$ the number of edges that connect P with Q.

Definition

*The **adjacency operator** A_X acting on the space of L^2-functions $f : X_0 \to \mathbb{C}$ is defined as*

$$A_X(f)(P) = \sum_{Q \sim P} \mu(P, Q) f(Q),$$

where we sum over adjacent vertices with the multiplicity of the number of edges linking them.

The adjacency operator commutes with the induced right action of the group of graph automorphisms on $L^2(X_0)$.

Adjacency operators for graphs and Ramanujan graphs

Let X be a finite graph of constant valency $q+1$. The **trivial eigenvalues** of A_X acting on $L^2(X_0)$ are $\lambda = \pm(q+1)$. These are obtained by the constant non-zero function for $\lambda = q+1$, and by the 'alternating function' with $f(P) = -f(Q) \neq 0$ for all $P \sim Q$ for $\lambda = -(q+1)$. The latter only exists if X has a bipartite structure.

Adjacency operators for graphs and Ramanujan graphs

Alon and Boppana prove that asymptotically in families of finite $(q+1)$-regular graphs X_n with diameter tending to ∞ the largest absolute value of a non-trivial eigenvalue $\lambda(X_n)$ of the adjacency operator A_{X_n} has limes inferior

$$\underline{\lim}_{n\to\infty} \lambda(X_n) \geqslant 2\sqrt{q}.$$

This estimate motivates the definition as follows.

Definition

A finite $(q+1)$-regular graph X is defined to be a ***Ramanujan graph*** *if all non-trivial eigenvalues λ of the adjacency operator A_X have absolute value $\lambda \leqslant 2\sqrt{q}$.*

Higher-dimensional Ramanujan cube complexes

We write $P \sim_v Q$ if two vertices in the product of d trees are adjacent in v-direction, $v \in \{1, ..., d\}$.

Definition

We define an ***adjacency operator*** A_v ***in*** v***-direction*** *on* $L^2(G/K)$ *by*

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Definition

Let $X \to \Delta^d$ *be a finite cubical complex of dimension d that has constant valency* $q_v + 1$ *in all directions. Then X is a* ***cubical Ramanujan complex****, if for each* $v \in \{1, \ldots, d\}$*, the eigenvalues* λ *of* A_v *are trivial, i.e.,* $\lambda = \pm(q_v + 1)$*, or non-trivial and then bounded by*

$$\lambda \leqslant 2\sqrt{q_v}.$$

Higher-dimensional Ramanujan cube complexes

Theorem

Let $\Gamma \subseteq \Gamma_{M,\delta}$ be a congruence subgroup. Then the quotient X_Γ of a product of d trees by Γ is a cubical Ramanujan complex.

We conjecture that infinitely many of the Ramanujan complexes of the Theorem above are higher-dimensional coboundary expanders of bounded degree in the sense of Gromov.

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- The Cayley graph of Γ with respect to the generating set $\{a, b\}$, $Cay(\Gamma, \{a, b\})$, is a homogeneous tree of degree 4.
- The vertices of the tree are elements of Γ *i.e.* reduced words in $S = \{a, b, a^{-1}, b^{-1}\}$.

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- Ω has a natural compact (totally disconnected) topology :
- if $x \in \Gamma$ then let $\Omega(x)$ be all semi-infinite words with the prefix x
- $\Omega(x)$ is open and closed in Ω and the sets $g\Omega(x)$ and $g(\Omega \setminus \Omega(x))$, where $g \in \Gamma$ and $x \in S$, form a base for the topology of Ω.

Cuntz-Krieger algebras

Left multiplication by $x \in \Gamma$ induces an action α of Γ on $C(\Omega)$ by

$$\alpha(x)f(w) = f(x^{-1}w).$$

$C(\Omega) \rtimes \Gamma$ is generated by $C(\Omega)$ and the image of a unitary representation π of Γ
such that $\alpha(g)f = \pi(g)f\pi^*(g)$ for $f \in C(\Omega)$ and $g \in \Gamma$ and every such C^*-algebra is a quotient of $C(\Omega) \rtimes \Gamma$.

Cuntz-Krieger algebras

For $x \in \Gamma$, let p_x denote the projection defined by the characteristic function $\mathbf{1}_{\Omega(x)} \in C(\Omega)$.
For $g \in \Gamma$, we have

$$gp_xg^{-1} = \alpha(g)\mathbf{1}_{\Omega(x)} = \mathbf{1}_{g\Omega(x)}$$

and therefore for each $x \in S$,

$$p_x + xp_{x^{-1}}x^{-1} = \mathbf{1}.$$

$$p_a + p_{a^{-1}} + p_b + p_{b^{-1}} = \mathbf{1}$$

Partial isometries

For $x \in S$ we define a *partial isometry* $s_x \in C(\Omega) \rtimes \Gamma$ by

$$s_x = x(\mathbf{1} - p_{x^{-1}}).$$

Then,

$$s_x s_x^* = x(\mathbf{1} - p_x)x^{-1} = p_x$$

and

$$s_x^* s_x = \mathbf{1} - p_{x^{-1}} = \sum_{y \neq x^{-1}} s_y s_y^*.$$

These relations show that the partial isometries s_x, for $x \in S$, generate the Cuntz-Krieger algebra $\mathcal{O}_A$.

Transition matrix

Where

$$A = \begin{pmatrix} \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix}$$

relative to $\{a, a^{-1}, b, b^{-1}\} \times \{a, a^{-1}, b, b^{-1}\}$.

Cuntz-Krieger algebras

Cuntz-Krieger (1980) constructed a C^*-algebra from a matrix $A = (A(i,j))_{i,j\in\Sigma}$, Σ a finite set, $A(i,j) \in \{0,1\}$ and where every row and every column of A is non-zero. A C^*-algebra is generated by partial isometries $S_i \neq 0$ $(i \in \Sigma)$ that act on a Hilbert space in such a way that their support projections $Q_i = S_i^* S_i$ and their range projections $P_i = S_i S_i^*$ satisfy the relations

$$P_i P_j = 0 \ (i \neq j), \ Q_i = \sum_{j\in\Sigma} A(i,j) P_j \ (i,j \in \Sigma). \tag{1}$$

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- Γ is a fundamental group of a polyhedron P defined earlier.
- If Γ be a group of type rotating automorphisms of a building Δ, then the C^*-algebra $C(\Lambda) \rtimes \Gamma$ is isomorphic to a higher rank Cuntz–Krieger algebra O_{A_1,A_2}.

nD polyhedral algebras

Higher rank generalizations of Cuntz–Krieger algebras, associated to a finite collection of transition matrices $A_j, j = 1, \ldots, r$, with entries in $\{0, 1\}$, associated to shifts in r different directions, with the transition matrices satisfying compatibility conditions, induced by the structure of the building.

The matrices give admissibility conditions for r-dimensional words in an assigned alphabet.

Polyhedral algebra: the alphabet is induced by a polygonal presentation.

nD polyhedral algebras

The following definition is inspired by works of Kumjian, Pask, Robertson, Steger, Sims in higher rank C*-algebras setting.

Definition

A *nD* polyhedral algebra is the universal $C^\star$-algebra generated by partial isometries $S_{u,v}$, where u and v are words in the given *nD* alphabet, with $t(u) = t(v)$, satisfying the relations

$$\begin{aligned} S^*_{u,v} &= S_{v,u} & S_{u,v}S_{v,w} &= S_{u,w} \\ S_{u,v} &= \sum S_{uw,vw} & S_{u,u}S_{v,v} &= 0, \ \forall u \neq v \end{aligned} \tag{2}$$

(The sum here is over n-dimensional words w with $o(w) = t(u) = t(v)$ and with shape $\sigma(w) = e_j$, for $j = 1, \ldots, n$, where e_j is the j-th standard basis vector in $\mathbb{Z}^n$.)

Theorem (J.Konter,AV)

The order of the class [$\mathbf{1}$] of the identity element $\mathbf{1}$ of $C(\Omega) \rtimes \Gamma$ in $K_0(C(\Omega) \rtimes \Gamma)$ is $q-1$, where Γ is a group acting on a triangular Euclidean building with three orbits and $q = 2^{2l-1}, l \in \mathbb{Z}$.

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- New applications of polygonal presentations to algebraic geometry: Beauville surfaces and fake quadrics.