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Simple, unital, separable UCT C^*-algebras with finite decomposition rank are classifiable

Abstract. Following on their recent TA (point-line) algebra classification, Gong, Lin, and Niu, together with me, have shown that the class of the title coincides with their class. By work of Niu, Santiago, Tikuisis, and me (together with earlier results) it follows that this class also coincides with the class of separable simple unital ASH algebras with slow dimension growth.

TA something algebras (cf. Lin's TAF)

The following result of Winter (2013) is important for showing that the Gong-Lin-Niu classifiable class coincides with the more concrete class of simple, unital, separable, Jiang-Su stable ASH algebras on the one hand, and the more abstract class of simple, unital, separable, finite nuclear dimension, ~~amenable (= nuclear),~~ ~~Jiang-Su stable,~~ finite, UCT, C^*-algebras such that every trace is quasidiagonal — alternatively, with finite decomposition rank — on the other hand.

(Recall that the Gong-Lin-Niu class is those algebras (assumed to be UCT) for which the tensor product with the universal UHF algebra Q is TA point-line — more specifically TA point-line with $K_1 = 0$).)

Winter (2013): Let $\mathcal{S}$ be a class of separable, unital C^*-algebras which can be finitely presented with weakly stable relations. Suppose further that $\mathcal{S}$ is closed under taking finite direct sums and under taking tensor products with finite dimensional C^*-algebras, and that $\mathcal{S}$ contains $\mathbb{C}$ (and therefore all finite dimensional C^*-algebras).

Let A be a simple, unital, separable, exact C^*-algebra with $\dim_{nuc} A < +\infty$ and $T(A) \neq \emptyset$, and suppose that there exists a system of maps

$$A \xrightarrow{\sigma_i} B_i \xrightarrow{\rho_i} A$$

$\boxed{\text{$\rho_i$ might as well be unital — replace A by $A \otimes Q$, and scale.}}$

such that

(i) $B_i \in \mathcal{S}$, $i \in \mathbb{N}$

(ii) ρ_i is an embedding, $i \in \mathbb{N}$

(iii) σ_i is c.p.c., $i \in \mathbb{N}$

(iv) $\bar{\sigma}: A \to \prod B_i / \bigoplus B_i$ induced by the sequence (σ_i) is a unital $*$-homomorphism

(v) $\sup\{|\tau(\rho_i\sigma_i(a) - a)| \mid \tau \in T(A)\} \to 0$, $a \in A$.

It follows that $A \otimes Q \in TA\mathcal{S}$. ($Q$ universal UHF)

The reduction (of either the abstract or the ASH class—as a special case by ENST) to the GLN TA (point-line) class, then, goes as follows.

Let A be simple, unital, separable, UCT, and have finite nuclear dimension (implies amenable (= nuclear) and Jiang-Su stable). The following three properties are equivalent.

(i) Every trace on A is quasidiagonal.

(ii) There exists an approximate factorization of traces through a completely positive unital approximate homomorphism from $A \otimes Q$ into a subalgebra which is a point-line algebra tensor Q. (Note unique stable C^*-algebra with ~~primitive~~ quotients Q and primitive spectrum

...

.)

$$\begin{tikzcd} \cdot \arrow[rr, "id"] \arrow[dr] & & \cdot \\ & \cdot \arrow[ur] & \end{tikzcd}$$

(iii) $A \otimes Q$ is TA (point-line).

(i) $\Rightarrow$ (iii) is Winter (2013).
(i) $\Rightarrow$ (ii) is EN (special case) and EGLN. (June 2015)
((iii) $\Rightarrow$ (i) is obvious, as consequence of Arveson extension theorem. (iii) $\Rightarrow$ (ii) it seems not obvious.)

Once have (iii), $A \otimes Q$ is classifiable by GLN, and hence A ($\cong A \otimes Z$) is classifiable by the one-parameter deformation isomorphism theorem of Winter (2014) (with additional work by Lin (2014—appendix to Winter) and Lin – Niu (2008).

To get ASH use ENST to get finite decomposition rank (which implies finite nuclear dimension and traces quasidiagonal). (Of course need to assume Jiang-Su stability, or slow dimension growth which by work of Toms and of Winter using the Cuntz semigroup implies this.) (Proved differently earlier also by EGNL.)

Also ENST implies that if A is the C^*-algebra associated with a minimal homeomorphism of a compact metrizable space, then $A \otimes Q$ has finite decomposition rank (because locally SH), and so (by EGLN) $A \otimes Z$ belongs to the GNL classifiable class. (This was proven earlier by Lin—using ENST.) (Also EGLN ASH.) (Hence by EN if mean dimension zero then A classifiable.)

Outline (rough!) of the proof of (i) $\Rightarrow$ (ii). (I.e., traces quasidiagonal implies traces factorize approximately, uniformly, via approximate homomorphisms, through ~~a~~ point-line algebras.)

1. Reduce to the situation that the identity on traces and K_0 factorizes approximately through a point-line algebra (with $K_1 = 0$) which might as well be taken as a subalgebra.

After tensoring with Q, have a Q-stabilized point-line algebra (more convenient). Say C.

2. First consider the case the primitive spectrum of C is just $[-\infty, +\infty]$. Assume we have an approximate homomorphism then $A \to \cancel{Q}\, C$ which is approximately compatible with the trace and K_0 maps. (In other words, a small adjustment of K_0 may be needed.)

3. Next, using (again!) quasidiagonality of traces, choose an approximate homomorphism (unital, completely positive) into the fibre at each point at infinity, again approximately compatible with respect to traces and K_0 (and the trace-K_0 map).

4. Using a general lemma, move the goalposts (the target K_0-classes) so that a slightly modified try – for the maps – exactly hits the K_0 goalposts.

5. Using the same general lemma, get conclusion of ~~Step~~ 2 for each line in the spectrum.

6. Using the same general lemma, modify each of the maps in 5 and each of the point maps in 4 so that, at the level of K_0 and traces, they are exactly compatible.

7. Now all that has to be done is glue the maps to the points at infinity – ~~brought~~ down to the ends of the various lines, to the ~~[illegible]~~ maps at the ends of these lines. Again, this can be done using the same general lemma – or in fact just (roughly speaking) Step 2. One connects these two maps, which have the same K_0 and traces, by a path, which does not move much at the level of traces, and sticks this in.

Even briefer (middle of the night!) summary.

(a) Get map $T(C_1) \to T(A)$, approximately compatible with $K_0(A) \to K_0(C_1)$.

(b) By a general lemma,

(first try at infinite points$)_{K_0}$ − (goalposts)

= (adjustment to goalposts!) − (adjustment to try$)_{K_0}$.

Moving subtracted terms to other side, and adding, and renormalizing, have

(second try at infinite goalposts$)_{K_0}$ = (new goalposts).

(c) Look at results of current try at infinite points at each end of each line — same K_0 at each end. Make initial try at each of closely spaced points (using quasidiagonality of traces) along each line. By another lemma,

(results on right, resp. left (same), of current try at infinite points$)_{K_0}$

− (tries at other points on lines$)_{K_0}$

= (adjustment of tries at other points$)_{K_0}$

− (results on right, resp. left (same), ends of new adjustment at infinite points$)_{K_0}$.

Moving terms, adding, and renormalizing as before, have K_0 of new tries at all chosen points of each line, together with K_0 of new (final) results at ~~each~~ both ends, equal.

(d) Now just glue. (Connect ends and close points along each line.)

General lemmas

I Basic (for everything). Let A be a unital separable simple amenable (= nuclear) quasidiagonal C^*-algebra satisfying the UCT. Assume $A \cong A \otimes Q$. Let a finite subset $G \subseteq A$ and $\epsilon_1, \epsilon_2 > 0$ be given. Let $p_1, p_2, \ldots, p_s \in \mathrm{Proj}_\infty(A)$ be projections such that $[1], [p_1], [p_2], \ldots, [p_s]$ are $\mathbb{Q}$-independent in $K_0 A$. Then there are a G-ϵ_1-multiplicative c.p.c. map $\sigma: A \to Q \otimes \mathcal{K}$ with $\sigma(1)$ a projection satisfying

$$\mathrm{tr}(\sigma(1)) < \epsilon_2,$$

and a $\delta > 0$, such that, for any $r_1, \ldots, r_s \in \mathbb{Q}$ with

$$|r_i| < \delta, \quad i = 1, \ldots, s,$$

there is a G-ϵ_1-multiplicative c.p.c. map $\mu: A \to Q \otimes \mathcal{K}$ such that $\mu(1) = \sigma(1)$ and

$$[\sigma(p_i)] - [\mu(p_i)] = r_i, \quad i = 1, \ldots, s.$$

(Proof uses all hypotheses.)

II (for (b) — moving goalposts at infinity). Let A be as in I. Let $G \subseteq A$, $\epsilon_1, \epsilon_2 > 0$, and $p_1, p_2, \cdots, p_s \in \mathrm{Proj}_\infty(A)$ be as in I. Then there is $\delta > 0$ with the following property.

Let $\psi_0, \psi_1 : Q^l \to Q^n$ be two unital homomorphisms, and set

$$\mathbb{D} = \{ x \in \mathbb{Q}^l ; (\psi_0)_*(x) = (\psi_1)_*(x) \} \subseteq \mathbb{Q}^l .$$

There exists a G-ϵ_1-multiplicative c.p.c. map

$$\Sigma : A \to Q^l$$

satisfying

$$\text{traces}\{[\Sigma(1_A)], [\psi_0 \circ \Sigma(1_A)], [\psi_1 \circ \Sigma(1_A)]\} < \epsilon_2, \text{ and}$$

$$[\Sigma(1_A)], \ [\Sigma(p_i)] \in \mathbb{D}, \quad i = 1, \cdots, s,$$

such that, for any $r_1, r_2, \cdots, r_s \in \mathbb{Q}^l$ with

$$\|r_i\|_\infty < \delta, \quad i = 1, \cdots, s,$$

there is a G-ϵ_1-multiplicative c.p.c. map $\mu : A \to Q^l$ such that $[\mu(1_A)] = [\Sigma(1_A)]$ and

$$[\Sigma(p_i)] - [\mu(p_i)] = r_i, \quad i = 1, \cdots, s.$$

$\underline{\text{III}}$ (for (c) — matching infinite points with lines).

Same as $\underline{\text{II}}$, except that now $r_1, r_2, \cdots, r_s \in \mathbb{Q}^n$, μ maps A to $\mathbb{Q}^n$,

$$[\psi_0 \circ \Sigma(1_A)] = [\psi_1 \circ \Sigma(1_A)] = [\mu(1_A)],$$

and

$$[\psi_0 \circ \Sigma(p_i)] - [\mu(p_i)] = [\psi_1 \circ \Sigma(p_i)] - [\mu(p_i)]$$
$$= r_i, \quad i = 1, 2, \cdots, s.$$

Important uniqueness lemma.

Let A be a simple, unital, separable $\mathcal{Z}$ C^*-algebra satisfying the UCT. For any finite subset $F \subseteq A$ and any $\epsilon > 0$, there exist $n \in \mathbb{N}$, a finite subset $G \subseteq A$, a finite subset $P \subseteq \mathrm{Proj}_\infty(A)$, and $\delta > 0$ with the following property.

For any three c.p.c. maps $\varphi, \psi, \xi : A \to Q$ which are G-δ-multiplicative, with

$$\varphi(1) = \psi(1) = 1_Q - \xi(1) \text{ a projection},$$

$$[\varphi(p)] = [\psi(p)] \text{ for all } p \in P, \text{ and}$$

$$\mathrm{tr}(\varphi(1)) = \mathrm{tr}(\psi(1)) < 1/n,$$

there exists a unitary $u \in Q$ such that

$$\| u^*(\varphi(a) \oplus \xi(a)) u - \psi(a) \oplus \xi(a) \| < \epsilon, \quad a \in F.$$