Von Neumann Algebras meet Quantum Information Theory

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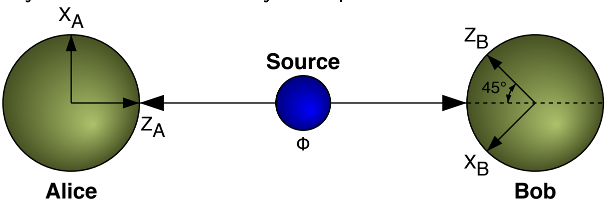
Alice and Bob, residing in spatially separated labs, each receives a quantum system on which they can perform measurements.



Let's say that Alice and Bob can measure any one of n possible observables each with k possible outcomes. Let

$$P(a, b \mid x, y)$$

be the probability that Alice gets outcome a and Bob outcome b, when Alice measures observable x and Bob measures observable y.

Definition: A PVM (projection valued measure) is a k-tuple $P_1, \ldots, P_k$ of projections on a Hilbert space H s.t. $\sum_{j=1}^{k} P_j = I_H$.

According to the mathematical model chosen for interpreting the physical separation of the labs (commutativity of observables, resp, tensor product), the correlation matrix $\left[P(a,b \mid x,y)\right] \in M_{nk}(\mathbb{R})$ belongs to one of the following convex sets:

$$\mathcal{C}_{qc} = \left\{ \left[\left\langle (P_a^x Q_b^y)\psi, \psi \right\rangle \right] : \forall x, y \, \{P_a^x\}_a, \{Q_b^y\}_b \, \mathrm{PVM}, [P_a^x, Q_b^y] = 0, \psi \in H \right\}$$

$$\mathcal{C}_{qs} = \left\{ \left[\left\langle (P_a^x \otimes Q_b^y)\psi, \psi \right\rangle \right] : \forall x, y \, \{P_a^x\}_a, \{Q_b^y\}_b \, \mathrm{PVM}, \ \psi \in H_A \otimes H_B \right\}.$$

Let $\mathcal{C}_{qa} = \mathrm{cl}(\mathcal{C}_{qs})$. Is the closure necessary? (Note: $\mathcal{C}_{qc}$ is closed.)

▶ $\mathcal{C}_{qa}$ is equal to the closure of the corresponding set, denoted $\mathcal{C}_q$, where only finite dimensional Hilbert spaces H_A, H_B are considered.

$$\mathcal{C}_q(n,k) \subseteq \mathcal{C}_{qs}(n,k) \subseteq \mathcal{C}_{qa}(n,k) \subseteq \mathcal{C}_{qc}(n,k).$$

Conjecture (Tsirelson): $\mathcal{C}_{qa}(n,k) = \mathcal{C}_{qc}(n,k)$.

Theorem (Kirchberg '93, Fritz/Junge et al.'10, Ozawa '12):

The following are equivalent:

1. *Tsirelson's conjecture is true:* $\mathcal{C}_{qc}(n,k) = \mathcal{C}_{qa}(n,k)$, $\forall n, k \geq 2$,
2. $C^*(\Gamma) \otimes_{\max} C^*(\Gamma) = C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$, $\forall n, k \geq 2$, *where* $\Gamma = \mathbb{Z}_k * \cdots * \mathbb{Z}_k$ *(n factors)*,
3. $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$,
4. *The Connes embedding problem has a positive answer.*

Recall that $\mathcal{C}_{qa} = \mathrm{cl}(\mathcal{C}_{qs})$, where

$$\mathcal{C}_{qs} = \left\{ \left[\langle (P^x_a \otimes Q^y_b)\psi, \psi \rangle \right] : \forall x, y \{P^x_a\}_a, \{Q^y_b\}_b \, \text{PVM}, \, \psi \in H_A \otimes H_B \right\},$$

$$\mathcal{C}_{qc} = \left\{ \left[\langle (P^x_a Q^y_b)\psi, \psi \rangle \right] : \forall x, y \{P^x_a\}_a, \{Q^y_b\}_b \, \text{PVM}, [P^x_a, Q^y_b] = 0, \psi \in H \right\}$$

Slofstra '16: $\mathcal{C}_{qs} \neq \mathcal{C}_{qc}$; March **'17**: $\mathcal{C}_{qs}$ not closed, for n, k large.

Dykema-Paulsen-Prakash Nov **'17**: $\mathcal{C}_q(5,2)$, $\mathcal{C}_{qs}(5,2)$ not closed.

Given unital C^*-algebras A and B, consider the following sets:

$$S_{\max} = \{\text{states on } A \otimes_{\max} B\},$$

$$S_{\min} = \{\text{states on } A \otimes_{\max} B \text{ that factor through } A \otimes_{\min} B\},$$

$$S_{\rm sp} = \{\varphi \in S_{\max} : \exists \pi_A \colon A \to B(H_A),\ \pi_B \colon B \to B(H_B) * -\text{rep}'\text{s},$$
$$\exists \psi \in H_A \otimes H_B : \varphi(x) = \langle (\pi_A \otimes \pi_B)(x)\psi, \psi\rangle\ \forall x\}.$$

- $S_{\rm sp} \subseteq S_{\min} \subseteq S_{\max}$; and all three sets are convex.
- $S_{\rm sp}$ is weak*-dense in $S_{\min}$; and $S_{\min}$ and $S_{\max}$ are closed.
- $S_{\min} = S_{\max}$ iff $A \otimes_{\max} B = A \otimes_{\min} B$.

Problems: For which C^*-algebras A and B is $S_{\rm sp} \neq S_{\min}$?
In specific examples, find concrete states that witness "$\neq$".
How can one see that a state in $S_{\min}$ does not belong to $S_{\rm sp}$?

Let $A = B = C^*(\Gamma)$, where $\Gamma = \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$ (n free factors).
Set $M = \mathrm{span}\{e_a^x \otimes e_b^y : a, b, x, y\} \subseteq C^*(\Gamma) \otimes_{\max} C^*(\Gamma)$ (fin. dim.)

$$\begin{aligned}
\mathcal{C}_{qs} &= \{[\varphi(e_a^x \otimes e_b^y)]_{a,b,x,y} : \varphi \in S_{\mathrm{sp}}\} = S_{\mathrm{sp}}|_M, \\
\mathcal{C}_{qa} &= \{[\varphi(e_a^x \otimes e_b^y)]_{a,b,x,y} : \varphi \in S_{\min}\} = S_{\min}|_M, \\
\mathcal{C}_{qc} &= \{[\varphi(e_a^x \otimes e_b^y)]_{a,b,x,y} : \varphi \in S_{\max}\} = S_{\max}|_M.
\end{aligned}$$

Slofstra '16: $\mathcal{C}_{qs} \neq \mathcal{C}_{qc}$.
Slofstra '17: $\mathcal{C}_{qs}$ is not closed, so $\mathcal{C}_{qa} \neq \mathcal{C}_{qs}$ and $S_{\min} \neq S_{\mathrm{sp}}$.

- **Brown-M-Rørdam:** Concrete example of $\omega \in S_{\max}$, arising from the "left-right" repres. of Γ, that does not belong to S_{sp}. (The proof relies in part on **Akemann-Ostrand '74**.)

However, $\omega|_M$ does belong to $S_{\mathrm{sp}}|_M$.

Recall, we do not know if $S_{\max} = S_{\min}$ (this is equivalent to the Connes embedding problem).

Consider following sets of $n \times n$ matrices of unitary correl, $n \geq 2$:

$$\mathcal{F}_{\rm matr}(n) = \bigcup_{k \geq 1} \left\{ \left[{\rm tr}_k(u_j^* u_i) \right] : u_1, \ldots, u_n \text{ unitaries in } M_k(\mathbb{C}) \right\},$$

$$\mathcal{G}(n) = \left\{ \left[\tau(u_j^* u_i) \right] : u_1, \ldots, u_n \text{ unitaries in } (M, \tau) \right\},$$

$$\mathcal{F}_{\rm fin}(n) = \left\{ \left[\tau(u_j^* u_i) \right] : u_1, \ldots, u_n \text{ unitaries in } (\mathcal{A}, \tau) \right\}, \text{ where}$$

(M, τ) = finite von Neumann alg., $(\mathcal{A}, \tau)$ = finite dim C^*-alg.

- $\mathcal{F}_{\rm matr}(2) = \mathcal{G}(2)$ is closed. We will later show that $\mathcal{F}_{\rm matr}(n)$ is neither convex, nor closed, $\forall n \geq 3$, by a trick originating in ideas of **Regev-Slofstra-Vidick**. Moreover, $\forall n \geq 2$,

$${\rm conv}(\mathcal{F}_{\rm matr}(n)) = \mathcal{F}_{\rm fin}(n) \quad \text{and} \quad {\rm cl}(\mathcal{F}_{\rm matr}(n)) = {\rm cl}(\mathcal{F}_{\rm fin}(n)) := \mathcal{F}(n).$$

By a refinement of **Kirchberg '93**, **Dykema–Juschenko '09**: The Connes embedding problem has a positive answer iff

$$\mathcal{G}(n) = \mathcal{F}(n), \quad \forall n \geq 3.$$

For $n \geq 2$, consider now the following sets of $n \times n$ matrices of correlations arising from projections:

$$\mathcal{D}(n) = \Big\{ \big[\tau(p_j p_i)\big] : p_1, \dots, p_n \text{ projections in } (M, \tau) \Big\},$$
$$\mathcal{D}_{\mathrm{fin}}(n) = \Big\{ \big[\tau(p_j p_i)\big] : p_1, \dots, p_n \text{ projections in } (\mathcal{A}, \tau) \Big\},$$

(M, τ) = finite von Neumann alg. $(\mathcal{A}, \tau)$ = finite dim. C^*-algebra.

► For $n \geq 2$, $\mathcal{D}(n)$ is closed and convex, and $\mathcal{D}_{\mathrm{fin}}(n)$ is convex. Also, $\mathcal{D}_{\mathrm{fin}}(2) = \mathcal{D}(2)$. Not known if $\mathcal{D}_{\mathrm{fin}}(3)$, $\mathcal{D}_{\mathrm{fin}}(4)$ are closed.

Note: The Connes embedding problem has a positive answer iff $\mathcal{D}(n) = \mathrm{cl}(\mathcal{D}_{\mathrm{fin}}(n))$, $\forall n \geq 3$.

Theorem (M–Rørdam, June '18): $\mathcal{D}_{\mathrm{fin}}(n)$ not closed, $\forall n \geq 5$.

The proof follows ideas from **Dykema-Paulsen-Prakash**, but avoids graph correlation functions (and quantum games).

Projections adding up to a scalar multiple of the identity operator:

Let Σ_n be the set of $\alpha > 0$ for which $\exists$ projections $p_1, \ldots, p_n$ on a Hilbert space H such that $\sum_{j=1}^n p_j = \alpha \cdot I_H$.

- It is known that Σ_n is discrete, when $n \leq 4$.

Theorem (**Kruglyak-Rabanovich-Samoilenko '02**): Let $n \geq 5$. There exist projections $p_1, \ldots, p_n$ on a *finite dimensional* Hilbert space H so that $\sum_{j=1}^n p_j = \alpha \cdot I_H$ if and only if $\alpha \in \Sigma_n \cap \mathbb{Q}$. Furthermore,

$$\left[\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})\right] \subseteq \Sigma_n.$$

Note: The "only if" part is easy (with Tr standard trace on $B(H)$):

$$\sum_{j=1}^n p_j = \alpha \cdot I_H \implies \alpha \cdot \dim(H) = \sum_{j=1}^n \mathrm{Tr}(p_j).$$

For $n \geq 2$ and $1/n \leq t \leq 1$, consider the following $n \times n$ matrix:

$$A_t^{(n)}(i,j) = \begin{cases} t, & i = j, \\ \dfrac{t(nt-1)}{n-1}, & i \neq j. \end{cases}$$

Proposition: Let $(\mathcal{A}, \tau)$ be a unital C^*-alg with faithful tracial state τ, and $p_1, \ldots, p_n \in \mathcal{A}$ be projections. Set $\alpha = nt$.

- If

$$\tau(p_j p_i) = A_t^{(n)}(i,j), \qquad 1 \leq i,j \leq n,$$

then $\sum_{j=1}^n p_j = \alpha \cdot 1_{\mathcal{A}}$. Moreover, if $t \notin \mathbb{Q}$, then $\dim(\mathcal{A}) = \infty$.

- Respectively, if $\sum_{j=1}^n p_j = \alpha \cdot 1_{\mathcal{A}}$, then $\exists m \geq 1$ and projections $\widetilde{p}_1, \ldots, \widetilde{p}_n \in M_m(\mathcal{A})$ such that

$$(\tau \otimes \mathrm{tr}_m)(\widetilde{p}_j \widetilde{p}_i) = A_t^{(n)}(i,j), \qquad 1 \leq i,j \leq n.$$

Recall

$$A_t^{(n)}(i,j) = \begin{cases} t, & i = j, \\ \frac{t(nt-1)}{n-1}, & i \neq j. \end{cases}$$

Combining the previous proposition with the theorem of Kruglyak, Rabanovich and Samoilenko, we get

Theorem: Let $n \geq 5$, $t \in [\frac{1}{2}(1 - \sqrt{1 - 4/n}), \frac{1}{2}(1 + \sqrt{1 - 4/n})]$.

- If $t \in \mathbb{Q}$, then $A_t^{(n)} \in \mathcal{D}_{\mathrm{fin}}(n)$.
- If $t \notin \mathbb{Q}$, then $A_t^{(n)} \in \mathrm{cl}(\mathcal{D}_{\mathrm{fin}}(n)) \setminus \mathcal{D}_{\mathrm{fin}}(n)$.

In particular, $\mathcal{D}_{\mathrm{fin}}(n)$ is non-closed, when $n \geq 5$.

Theorem (M-Rørdam): $\mathcal{D}_{\mathrm{fin}}(n)$ is not closed, for all $n \geq 5$.

Some applications:

- $\mathcal{D}_{\mathrm{fin}}(n)$ not closed $\overset{(*)}{\Longrightarrow}$ $\mathcal{C}_q^s(n,2)$ (the set of *synchronous* quantum correlations) not closed $\Longrightarrow$ $\mathcal{C}_q(n,2)$ not closed.

$(*)$ uses (**Paulsen-Severini-Stahlke-Todorov-Winter '16**) result, asserting that $\mathcal{C}_q^s(n,2) = \mathcal{D}_q^s(n,2), \forall n \geq 2$.

- $\mathcal{D}_{\mathrm{fin}}(n)$ not closed $\Longrightarrow$ $\mathcal{F}_{\mathrm{fin}}(2n+1)$ not closed (again, using the trick originating in **Regev-Slofstra-Vidick**). This implies that $\mathcal{F}_{\mathrm{fin}}(n)$ is not closed, $\forall n \geq 11$.

This last non-closure result has unravelled some interesting infinite dimensional phenomena in Quantum Information Theory.