On the topological structure of the set of composition operators

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Dirichlet series and operator theory
Let $\mathcal{H}$ be the (Hilbert) Hardy space of Dirichlet series:

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\mathcal{H} = \left\{ \sum_{n \geq 1} a_n n^{-s} : \|f\|_{\mathcal{H}}^2 = \sum_{n \geq 1} |a_n|^2 < +\infty \right\}.
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Let also $\mathcal{G}$ be the set of symbols $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ inducing a bounded composition operator $C_\varphi$ on $\mathcal{H}$. 
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Recall (Gordon - Hedenmalm) that $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ belongs to $\mathcal{G}$ if and only if $\varphi(s) = c_0 s + \psi(s)$, where $c_0$ is a non-negative integer (called the characteristic of $\varphi$, i.e., $\text{char}(\varphi) = c_0$), and $\psi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges uniformly in $\mathbb{C}_\epsilon$ for every $\epsilon > 0$ and has the following properties:

(a) If $c_0 = 0$, then $\psi(\mathbb{C}_0) \subseteq \mathbb{C}_{1/2}$.
(b) If $c_0 \geq 1$, then either $\psi \equiv 0$ or $\psi(\mathbb{C}_0) \subseteq \mathbb{C}_0$. 
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We shall denote by $\mathcal{C}(\mathcal{H}) = \{ C_\varphi : \varphi \in \mathcal{G} \} \subset \mathcal{L}(\mathcal{H})$. 
Main question for this talk

What can be said about the topological structure of $C(\mathcal{H})$?

First result on the unit disc.
Theorem (Shapiro - Sundberg (1990))
The set of compact composition operators is an arcwise connected set in $C(\mathcal{H}_2(D))$.

The characteristic of a symbol (which is an integer) prevents this result from extending to $\mathcal{H}$.
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What can be said about the topological structure of $C(\mathcal{H})$? In particular,

- When do two composition operators $C_{\varphi_1}$ and $C_{\varphi_2}$ belong to the same component of $C(\mathcal{H})$?
- When are two composition operators $C_{\varphi_1}$ and $C_{\varphi_2}$ equal modulo some compact operator?
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The characteristic of a symbol (which is an integer) prevents this result from extending to $\mathcal{H}$. 
Proposition

The map $\mathcal{C}(\mathcal{H}) \to \mathbb{N}_0$, $\phi \mapsto \text{char}(\phi)$ is continuous.

Proof.

Let $(\phi_k)_k$, $\phi \in \mathcal{G}$ such that $C_{\phi_k} \to C_{\phi}$ and assume that $\text{char}(\phi_k) \neq \text{char}(\phi)$ for all $k$. 
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First case : $\text{char}(\varphi) = 0$.

Let $K_s$ be the reproducing kernel at $s \in \mathbb{C}_{1/2}$. Then $C_{\varphi}^*(K_s) = K_{\varphi(s)}$. 


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Let $K_s$ be the reproducing kernel at $s \in \mathbb{C}_{1/2}$. Then $C^*_\varphi(K_s) = K_{\varphi(s)}$.

When $\sigma \rightarrow +\infty$ one has $\varphi(\sigma) \rightarrow c_1$ and $\Re(\varphi_k(\sigma)) \rightarrow +\infty$ for $k \geq 1$. 
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When $\sigma \to +\infty$ one has $\varphi(\sigma) \to c_1$ and $\Re(\varphi_k(\sigma)) \to +\infty$ for $k \geq 1$.

Hence, $K_{\varphi(\sigma)} \to K_{c_1}$ whereas $K_{\varphi_k(\sigma)} \to 1$. Therefore,

$$
\| C_{\varphi} - C_{\varphi_k} \| = \| C^*_\varphi - C^*_\varphi_k \|
\geq \limsup_{\sigma \to +\infty} \| C^*_\varphi(K_\sigma) - C^*_\varphi_k(K_\sigma) \| / \| K_\sigma \|
\geq \limsup_{\sigma \to +\infty} \| K_{\varphi(\sigma)} - K_{\varphi_k(\sigma)} \|
\geq \| K_{c_1} - 1 \| > 0.
$$
Second case: $\text{char}(\varphi) = c_0 > 0$ and $\text{char}(\varphi_k) > c_0$ for an infinite number of integers $k$. 
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Write \( \varphi(s) = c_0 s + c_1 + \sum_{n \geq 2} c_n n^{-s} \). Then

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2^{-\varphi(s)} = 2^{-c_0 s} 2^{-c_1} 2^{-\sum_{n \geq 2} c_n n^{-s}}
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= 2^{-c_0 s} 2^{-c_1} \prod_{n \geq 2} \left(1 + \sum_{k} d_{k,n} \left(n^k\right)^{-s}\right).
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In particular, writing \( 2^{-\varphi(s)} = \sum_{n \geq 1} a_n n^{-s} \), one has \( a_{2j} = 0 \) provided \( j < c_0 \) and \( a_{2c_0} = 2^{-c_1} \).
Second case: $\text{char}(\varphi) = c_0 > 0$ and $\text{char}(\varphi_k) > c_0$ for an infinite number of integers $k$. Write $\varphi(s) = c_0 s + c_1 + \sum_{n \geq 2} c_n n^{-s}$. Then

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In particular, writing $2^{-\varphi(s)} = \sum_{n \geq 1} a_n n^{-s}$, one has $a_{2j} = 0$ provided $j < c_0$ and $a_{2c_0} = 2^{-c_1}$.

Similarly, writing $2^{-\varphi_k(s)} = \sum_{n \geq 1} b_n(k) n^{-s}$, one has $b_{2c_0}(k) = 0$ for those $k$ such that $\text{char}(\varphi_k) > c_0$. This contradicts $b_{2c_0}(k) \to a_{2c_0}$. 
Second case: \( \text{char}(\varphi) = c_0 > 0 \) and \( \text{char}(\varphi_k) > c_0 \) for an infinite number of integers \( k \).

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Third case: \( \text{char}(\varphi) = c_0 > 0 \) and \( \text{char}(\varphi_k) < c_0 \) for all \( k \).
Theorem

Let $c_0 \in \mathbb{N}_0$. The set of compact composition operators with characteristic equal to $c_0$ is arcwise connected.
$\mathbb{T}^\infty$ can be identified with the dual group of $(\mathbb{Q}_+, \cdot)$ by

$$z \in \mathbb{T}^\infty \mapsto \chi_z, \quad \chi_z(2) = z_1, \chi_z(3) = z_2, \ldots, \chi_z(p_m) = z_m.$$
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2. For $\chi \in \mathbb{T}^\infty$ and $f = \sum_n a_n n^{-s} \in \mathcal{H}$, define $f_\chi(s) = \sum_n a_n \chi(n) n^{-s}$. 


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**Proposition**

Let $f, F \in \mathcal{H}$. TFAE:

- There exists $(\tau_k) \subset \mathbb{R}$, $f(\cdot + i\tau_k) \to F$ uniformly on compact subsets of $\mathbb{C}_{1/2}$.
- There exists $\chi \in \mathbb{T}^\infty$, $F = f_\chi$. 
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Let $f \in \mathcal{H}$. For almost all $\chi \in \mathbb{T}^\infty$, $f_\chi$ converges in $\mathbb{C}_0$, $f_\chi(it) = \lim_{\sigma \to 0} f_\chi(\sigma + it)$ exists for almost all $t \in \mathbb{R}$ and
\[
\|f\|^2 = \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(it)|^2 d\mu(t) dm(\chi).
\]
Let $f \in \mathcal{H}$, $\varphi = c_0 s + \psi \in \mathcal{G}$. Define $\varphi_\chi = c_0 s + \psi_\chi$. Then

$$(f \circ \varphi)_\chi = f_{\chi c_0} \circ \varphi_\chi.$$
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Let $\varphi_0 = c_0s + \psi_0$, $\varphi_1 = c_0 + \psi_1$ be two compact composition operators on $\mathcal{H}$. 

Let $f \in \mathcal{H}$ and let us estimate $\|C\varphi_\lambda(f) - C\varphi_\lambda'(f)\|_2$.

$$\|C\varphi_\lambda(f) - C\varphi_\lambda'(f)\|_2 \leq \int \int |(f \circ \varphi_\lambda \chi)(it) - (f \circ \varphi_\lambda' \chi)(it)|^2 d\mu(t) dm(\chi).$$

Assume that $\varphi_0(C^+)$, $\varphi_1(C^+) \subset C^a$ for some $a > 1/2$. Then $(\varphi_\lambda \chi)(it) \in C^a$ for all $\chi$ and all $\lambda$ so that $\|f \chi c_0 \circ (\varphi_\lambda \chi)(it) - f \chi c_0 \circ (\varphi_\lambda' \chi)(it)\|_2 \leq C_a \|f\|_2 |(\varphi_\lambda \chi)(it) - (\varphi_\lambda' \chi)(it)|^2.$
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Let \( \varphi_0 = c_0 s + \psi_0 \), \( \varphi_1 = c_0 + \psi_1 \) be two compact composition operators on \( \mathcal{H} \). Define \( \varphi_\lambda = c_0 s + (1 - \lambda)\psi_0 + \lambda \psi_1 \). Is the map \( \lambda \mapsto C_{\varphi_\lambda} \in \mathcal{C}(\mathcal{H}) \) continuous? Is each \( C_{\varphi_\lambda} \) compact?
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Fix $f \in \mathcal{H}$ and let us estimate $\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\|$. 

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\leq \int \int |f_{\chi_0} \circ (\varphi_\lambda)(it) - f_{\chi_0} \circ (\varphi_{\lambda'})(it)|^2 d\mu dm.
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\leq \int \int \| f_{\chi c_0} \circ (\varphi_\lambda) \chi(it) - f_{\chi c_0} \circ (\varphi_{\lambda'}) \chi(it) \|^2 d\mu dm.
\]

Assume that \( \varphi_0(\mathbb{C}_+) \), \( \varphi_1(\mathbb{C}_+) \) \( \subset \mathbb{C}_a \) for some \( a > 1/2 \). Then \( (\varphi_\lambda) \chi(it) \in \overline{\mathbb{C}_a} \) for all \( \chi \) and all \( \lambda \) so that

\[
| f_{\chi c_0} \circ (\varphi_\lambda) \chi(it) - f_{\chi c_0} \circ (\varphi_{\lambda'}) \chi(it) \|^2 \leq C_a \| f \|^2 |(\varphi_\lambda) \chi(it) - (\varphi_{\lambda'}) \chi(it) \|^2.
\]
Recall that \( \varphi_\lambda = c_0 s + (1 - \lambda) \psi_0 + \lambda \psi_1 \). Then

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(\varphi_\lambda)_\chi(it) - (\varphi_{\lambda'})_\chi(it) = (\lambda - \lambda'((\psi_0)_\chi(it) - (\psi_1)_\chi(it)).
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Recall that $\varphi_\lambda = c_0 s + (1 - \lambda) \psi_0 + \lambda \psi_1$. Then

$$(\varphi_\lambda)_\chi(it) - (\varphi_{\lambda'})_\chi(it) = (\lambda - \lambda')( (\psi_0)_\chi(it) - (\psi_1)_\chi(it)).$$

Therefore,

$$\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\|^2 \leq C_a |\lambda - \lambda'|^2 \|f\|^2 \int \int |(\psi_0)_\chi(it) - (\psi_1)_\chi(it)|^2 d\mu dm.$$
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If $|\psi_0|$ and $|\psi_1|$ are bounded, we finally find

$$\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\|^2 \leq M |\lambda - \lambda'|^2 \cdot \|f\|^2.$$
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Until now we have shown that, if $\varphi_0 = c_0 s + \psi_0$ and $\varphi_1 = c_0 s + \psi_1$ are such that
- $\varphi_0(\mathbb{C}_+), \varphi_1(\mathbb{C}_+) \subset \mathbb{C}_a$ for some $a > 1/2$;
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then there is a continuous arc of compact composition operators between $C\varphi_0$ and $C\varphi_1$. 

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Let $\varphi$ inducing a compact composition operator. Find a continuous arc of compact composition operators between $C\varphi$ and $C\tilde{\varphi}$ where $\tilde{\varphi}$ satisfies the above assumptions.
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Let $\varphi$ inducing a compact composition operator. Find a continuous arc of compact composition operators between $C\varphi$ and $C\tilde{\varphi}$ where $\tilde{\varphi}$ satisfies the above assumptions.

Consider for $\sigma \in [0, 1]$, $\varphi_\sigma = \varphi(\cdot + \sigma)$. Then $\varphi_1$ satisfies the above assumptions. Is the map $\sigma \in [0, 1] \mapsto C\varphi_\sigma$ continuous?
Let $\varphi \in \mathcal{G}$ such that $C_\varphi$ is compact and let $\varphi_\sigma = \varphi(\cdot + \sigma)$.

**Lemma**

*The map $\sigma \in [0, 1] \mapsto C_\varphi$ is continuous.*
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**Lemma**

*The map $\sigma \in [0, 1] \mapsto C_{\varphi_\sigma}$ is continuous.*

**Proof.**

For $\sigma \geq 0$ define $T_\sigma(f) = f(\cdot + \sigma)$, so that $C_{\varphi_\sigma} = T_\sigma \circ C_\varphi$. Now,
Let $\varphi \in G$ such that $C_{\varphi}$ is compact and let $\varphi_\sigma = \varphi(\cdot + \sigma)$.

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For $\sigma \geq 0$ define $T_\sigma(f) = f(\cdot + \sigma)$, so that $C_{\varphi_\sigma} = T_\sigma \circ C_{\varphi}$. Now,

1. For a fixed $g \in \mathcal{H}$, $T_\sigma(g) \to T_{\sigma_0}(g)$ as $\sigma \to \sigma_0$. 


Let $\varphi \in G$ such that $C_{\varphi}$ is compact and let $\varphi_\sigma = \varphi(\cdot + \sigma)$.

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For $\sigma \geq 0$ define $T_\sigma(f) = f(\cdot + \sigma)$, so that $C_{\varphi_\sigma} = T_\sigma \circ C_\varphi$. Now,

1. For a fixed $g \in \mathcal{H}$, $T_\sigma(g) \to T_{\sigma_0}(g)$ as $\sigma \to \sigma_0$.
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Let \( \varphi \in \mathcal{G} \) such that \( C_\varphi \) is compact and let \( \varphi_\sigma = \varphi(\cdot + \sigma) \).

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**Proof.**

For \( \sigma \geq 0 \) define \( T_\sigma(f) = f(\cdot + \sigma) \), so that \( C_{\varphi_\sigma} = T_\sigma \circ C_\varphi \). Now,

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2. The family \( \{T_\sigma : \sigma \in [0, 1]\} \) is equicontinuous.
3. The set \( \{C_\varphi(f) : \|f\| \leq 1\} \) has compact closure.
Let \( \varphi \in G \) such that \( C_\varphi \) is compact and let \( \varphi_\sigma = \varphi(\cdot + \sigma) \).

**Lemma**

*The map \( \sigma \in [0, 1] \mapsto C_{\varphi_\sigma} \) is continuous.*

**Proof.**

For \( \sigma \geq 0 \) define \( T_\sigma(f) = f(\cdot + \sigma) \), so that \( C_{\varphi_\sigma} = T_\sigma \circ C_\varphi \). Now,

1. For a fixed \( g \in \mathcal{H} \), \( T_\sigma(g) \to T_{\sigma_0}(g) \) as \( \sigma \to \sigma_0 \).
2. The family \( \{ T_\sigma : \sigma \in [0, 1] \} \) is equicontinuous.
3. The set \( \{ C_\varphi(f) : \|f\| \leq 1 \} \) has compact closure.

The lemma follows from a (standard) compactness argument.
Two general statements

**Theorem (Positive characteristic)**

Let $\varphi_0$ and $\varphi_1 \in G$ with $\text{char}(\varphi_0) = \text{char}(\varphi_1) =: c_0 \geq 1$ and write them $\varphi_0 = c_0 s + \psi_0$, $\varphi_1 = c_0 s + \psi_1$. Assume moreover that there exists $C > 0$ such that

- $|\varphi_0 - \varphi_1| \leq C \min(\Re\varphi_0, \Re\varphi_1)$;
- $|\psi_0|, |\psi_1| \leq C$;
- $|\psi'_0|, |\psi'_1| \leq C$.

Then $C\varphi_0$ and $C\varphi_1$ belong to the same component of $C(\mathcal{H})$. 

**Theorem (Zero characteristic)**

Let $\varphi_0$ and $\varphi_1 \in G$ with $\text{char}(\varphi_0) = \text{char}(\varphi_1) = 0$. Assume that there exists $C > 0$ such that

$|\varphi_0 - \varphi_1| \leq C \min(|\Re\varphi_0 - 1/2|/|1+\varphi_0|^2, |\Re\varphi_1 - 1/2|/|1+\varphi_1|^2)$.

Then $C\varphi_0$ and $C\varphi_1$ belong to the same component of $C(\mathcal{H})$. 

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Two general statements

Theorem (Positive characteristic)

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Then \( C\varphi_0 \) and \( C\varphi_1 \) belong to the same component of \( C(\mathcal{H}) \).

Theorem (Zero characteristic)

Let \( \varphi_0 \) and \( \varphi_1 \in \mathcal{G} \) with \( \text{char}(\varphi_0) = \text{char}(\varphi_1) = 0 \). Assume that there exists \( C > 0 \) such that \( |\varphi_0 - \varphi_1| \leq C \min \left( \frac{\Re\varphi_0 - 1/2}{|1+\varphi_0|^2}, \frac{\Re\varphi_1 - 1/2}{|1+\varphi_1|^2} \right) \). Then \( C\varphi_0 \) and \( C\varphi_1 \) belong to the same component of \( C(\mathcal{H}) \).
Idea for the proof

Let $\varphi_0, \varphi_1 \in \mathcal{G}$. As before, define $\varphi_\lambda = (1 - \lambda)\varphi_0 + \lambda\varphi_1$. Write

$$\|C_{\varphi_\lambda}(f) - C_{\varphi_\lambda'}(f)\|^2 \leq \int \int |f_{\chi_0}((\varphi_\lambda)_{\chi}(it)) - f_{\chi_0}((\varphi_\lambda')_{\chi}(it))|^2 d\mu dm.$$
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Observe that

$$f_{\chi_0}((\varphi_{\lambda})_{\chi}(it)) - f_{\chi_0}((\varphi_{\lambda}')_{\chi}(it)) = ((\varphi_1)_{\chi}(it) - (\varphi_0)_{\chi}(it)) \times \int_{\lambda}^{\lambda'} f'_{\chi_0}((\varphi_r)_{\chi}(it))dr.$$
Idea for the proof

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\]

By Jensen’s inequality,

\[
\|C_{\varphi_\lambda}(f) - C_{\varphi_\lambda'}(f)\|^2 \leq |\lambda' - \lambda|^2 \int_{T^\infty} \int_{\mathbb{R}} |(\varphi_1)\chi(it) - (\varphi_0)\chi(it)|^2 \times |f'_{\chi^0}((\varphi_r)\chi(it))|^2 d\mu(t) dm(\chi) dr.
\]
Therefore, it suffices to show that there exists $C \geq 1$ such that, for all $r \in [0, 1]$,

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One wants to prove that

$$\|((\varphi_1 - \varphi_0)C_{\varphi r}(f'))\|_{\mathcal{H}} \leq C \|f\|_{\mathcal{H}}.$$
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$$A = \left\{ f = \sum_n a_n n^{-s} : \|f\|_A^2 := \int_{T^\infty} \int_\mathbb{R} \int_0^1 |f_\chi(s)|^2 \sigma d\sigma d\mu dm < +\infty \right\}.$$

**Lemma**

$f \in H \iff f' \in A.$
Therefore, it suffices to show that there exists $C \geq 1$ such that, for all $r \in [0,1],$

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**Lemma**

$f \in \mathcal{H} \iff f' \in \mathcal{A}.$

Therefore, it suffices to prove that

$$\|(\varphi_1 - \varphi_0)C_{\varphi_r}(f')\|_H \leq C \|f'\|_\mathcal{A}.$$
Let $w : \mathbb{C}_{1/2} \to \mathbb{C}$ be a Dirichlet series, $\varphi \in \mathcal{G}$. When do $wC\varphi : f \mapsto w \cdot f \circ \varphi$ defines a bounded operator from $A$ to $H$?
General problem: boundedness of weighted composition operators from $A$ to $H$

Let $w : \mathbb{C}_{1/2} \to \mathbb{C}$ be a Dirichlet series, $\varphi \in G$. When do $wC_\varphi : f \mapsto w \cdot f \circ \varphi$ defines a bounded operator from $A$ to $H$?

The assumptions we have made give an answer when $w = \varphi_1 - \varphi_0$ and $\varphi = \varphi_r$. 
General problem: boundedness of weighted composition operators from $\mathcal{A}$ to $\mathcal{H}$

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Two different proofs:
- For $c_0 = 0$, we reduce to Hardy and Bergman spaces of the unit disc and use Carleson measures.
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Two different proofs:

- For $c_0 = 0$, we reduce to Hardy and Bergman spaces of the unit disc and use Carleson measures.
- For $c_0 \geq 1$, we work directly with Dirichlet series in $\mathcal{A}$ and in $\mathcal{H}$ and use some Nevanlinna counting functions.
Application 1: linear symbols

Let \((q_j)_{j=1,...,d}\) be multiplicatively independent positive integers and let
\[
\varphi_0 = c_0 s + c_1 + \sum_{j=1}^{d} c q_j^{-s} \in G, \ c_0 \geq 1.
\]
Application 1: linear symbols

Let \((q_j)_{j=1,...,d}\) be multiplicatively independent positive integers and let
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\varphi_0 = c_0 s + c_1 + \sum_{j=1}^{d} c_{q_j} q_j^{-s} \in \mathcal{G}, \quad c_0 \geq 1.
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For \(\delta > 0\) sufficiently small, define
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\varphi_1 = c_0 s + c_1 + \sum_{j=1}^{d} c_{q_j} q_j^{-s} + \delta \left( c_1 + \sum_{j=1}^{d} c_{q_j} q_j^{-s} \right)^2.
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\(C_{\varphi_0}\) and \(C_{\varphi_1}\) are in the same component.
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- \(C_{\varphi_0}\) is not compact
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This disproves a conjecture of Shapiro and Sundberg in this setting (already disproved on \(H^2(D)\) by Bourdon and by Moorhouse and Tonge).

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Application 2: coefficients of the Bohr lift

Let $\varphi(s) = c_0 s + \psi(s) \in G$, $\psi(s) = \sum_{n=1}^{N} c_n n^{-s}$ be a Dirichlet polynomial symbol with $c_0 \geq 1$. Define the Bohr lift $B\psi$ of $\psi$ by

$$B\psi(z) = \sum_{n=p_1^{\alpha_1} \cdots p_d^{\alpha_d} = 1}^{N} c_n z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$ 

Then $B\psi$ maps $\mathbb{D}^d$ into $\mathbb{C}_+$. 
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Then \( B\psi \) maps \( \mathbb{D}^d \) into \( \mathbb{C}_+ \).

Let \( \Gamma(B\psi) = \{ z \in \mathbb{T}^d : \Re(B\psi(z)) = 0 \} \).

**Definition**

Let \( z \in \Gamma(B\psi) \). We say that \( \varphi \) has Dirichlet contact of order \( n \) at \( z \) if there exists a neighbourhood \( \mathcal{U} \) of \( z \) in \( \mathbb{T}^d \) such that, for all \( w \in \mathcal{U} \),

\[
\left| \Im m(B\psi(w) - B\psi(z)) \right|^{2n} \lesssim \Re(B\psi(w)).
\]
Corollary

Let $\varphi_0, \varphi_1 \in \mathcal{G}$ be Dirichlet polynomial symbols with $\text{char}(\varphi_0) = \text{char}(\varphi_1) \geq 1$. Assume that $\Gamma(B\psi_0) = \Gamma(B\psi_1)$ and that, for all $z \in \Gamma(B\psi_0)$, there exists $n \in \mathbb{N}$ such that

- $B\psi_0(z) = B\psi_1(z)$;
- $\varphi_0$ and $\varphi_1$ have a Dirichlet contact of order $2n$ at $z$;
- for $|\alpha| \leq 2n - 1$, $\partial_\alpha B\psi_0(z) = \partial_\alpha B\psi_1(z)$.

Then $C_{\varphi_0}$ and $C_{\varphi_1}$ belong to the same component of $C(\mathcal{H})$. 
How to prove that two composition operators do not belong to the same component? In particular, what about $\phi_0(s) = s + 1 - 2^{-s}$ and $\phi_1(s) = s + 1 - 3^{-s}$?
Open questions and work in progress

1. How to prove that two composition operators do not belong to the same component? In particular, what about $\varphi_0(s) = s + 1 - 2^{-s}$ and $\varphi_1(s) = s + 1 - 3^{-s}$?

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2. Do there exist isolated composition operators on $\mathcal{H}$? (true on $H^2(\mathbb{D})$ by a result of Berkson)

3. Do the compact composition operators form a connected component of $\mathcal{C}(\mathcal{H})$? (false in $H^2(\mathbb{D})$ by a result of Gallardo, Gonzalez, Nieminen and Saksman)

4. Can we use these methods to give conditions implying that $C_{\varphi_0} - C_{\varphi_1}$ is compact?
Theorem

Let $\varphi_0$ and $\varphi_1 \in G$ with $\text{char}(\varphi_0) = \text{char}(\varphi_1) = 0$. Assume that

$$|\varphi_0 - \varphi_1| = o\left(\min\left(\frac{\Re \varphi_0 - 1/2}{|1 + \varphi_0|^2}, \frac{\Re \varphi_1 - 1/2}{|1 + \varphi_1|^2}\right)\right) \text{ as } \Re(s) \to 0.$$ 

Then $C\varphi_0 - C\varphi_1$ is compact.
**Theorem**

Let $\varphi_0$ and $\varphi_1 \in \mathcal{G}$ with $\text{char}(\varphi_0) = \text{char}(\varphi_1) = 0$. Assume that

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Then $C_{\varphi_0} - C_{\varphi_1}$ is compact.

What happens for $c_0 \geq 1$? For instance,

**Conjecture**

Let $\varphi_0, \varphi_1 \in \mathcal{G}$ be Dirichlet polynomial symbols with $\text{char}(\varphi_0) = \text{char}(\varphi_1) \geq 1$. Assume that $\Gamma(\mathcal{B}\psi_0) = \Gamma(\mathcal{B}\psi_1)$ and that, for all $z \in \Gamma(\mathcal{B}\psi_0)$, there exists $n \in \mathbb{N}$ such that

- $\mathcal{B}\psi_0(z) = \mathcal{B}\psi_1(z)$;
- $\varphi_0$ and $\varphi_1$ have a Dirichlet contact of order $2n$ at $z$;
- for $|\alpha| \leq 2n$, $\partial_{\alpha}\mathcal{B}\psi_0(z) = \partial_{\alpha}\mathcal{B}\psi_1(z)$.

Then $C_{\varphi_0} - C_{\varphi_1}$ is compact???
Frontiers of Operator Theory

CIRM (Marseille - Luminy)
29 November - 3 December 2021

See you soon there!
Thank you!