On the topological structure of the set of composition operators

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Dirichlet series and operator theory

Let $\mathcal{H}$ be the (Hilbert) Hardy space of Dirichlet series:

$$\mathcal{H} = \left\{ \sum_{n\geq 1} a_n n^{-s} : \ \|f\|_{\mathcal{H}}^2 = \sum_{n\geq 1} |a_n|^2 < +\infty \right\}.$$

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Recall (Gordon - Hedenmalm) that $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ belongs to $\mathcal{G}$ if and only if $\varphi(s) = c_0 s + \psi(s)$, where c_0 is a non-negative integer (called the *characteristic* of φ, i.e., $\mathrm{char}(\varphi) = c_0$), and $\psi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges uniformly in $\mathbb{C}_\epsilon$ for every $\epsilon > 0$ and has the following properties:

(a) If $c_0 = 0$, then $\psi(\mathbb{C}_0) \subseteq \mathbb{C}_{1/2}$.

(b) If $c_0 \geq 1$, then either $\psi \equiv 0$ or $\psi(\mathbb{C}_0) \subseteq \mathbb{C}_0$.

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We shall denote by $\mathcal{C}(\mathcal{H}) = \{C_\varphi : \ \varphi \in \mathcal{G}\} \subset \mathcal{L}(\mathcal{H})$.

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First result on the unit disc.

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The characteristic of a symbol (which is an integer) prevents this result from extending to $\mathcal{H}$.

Proposition

The map $\mathcal{C}(\mathcal{H}) \to \mathbb{N}_0$, $\varphi \mapsto \mathrm{char}(\varphi)$ *is continuous.*

Proof.

Let $(\varphi_k)_k,\ \varphi \in \mathcal{G}$ such that $C_{\varphi_k} \to C_\varphi$ and assume that $\mathrm{char}(\varphi_k) \neq \mathrm{char}(\varphi)$ for all k.

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First case : $\mathrm{char}(\varphi) = 0$.

Let K_s be the reproducing kernel at $s \in \mathbb{C}_{1/2}$. Then $C_\varphi^*(K_s) = K_{\varphi(s)}$.

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Hence, $K_{\varphi(\sigma)} \to K_{c_1}$ whereas $K_{\varphi_k(\sigma)} \to 1$. Therefore,

$$\begin{aligned}\|C_\varphi - C_{\varphi_k}\| &= \|C_\varphi^* - C_{\varphi_k}^*\| \\ &\geq \limsup_{\sigma\to+\infty} \|C_\varphi^*(K_\sigma) - C_{\varphi_k}^*(K_\sigma)\|/\|K_\sigma\| \\ &\geq \limsup_{\sigma\to+\infty} \|K_{\varphi(\sigma)} - K_{\varphi_k(\sigma)}\| \\ &\geq \|K_{c_1} - 1\| > 0.\end{aligned}$$

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In particular, writing $2^{-\varphi(s)} = \sum_{n\geq 1} a_n n^{-s}$, one has $a_{2^j} = 0$ provided $j < c_0$ and $a_{2^{c_0}} = 2^{-c_1}$.

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Similarly, writing $2^{-\varphi_k(s)}=\sum_{n\geq 1}b_n(k)n^{-s}$, one has $b_{2^{c_0}}(k)=0$ for those k such that $\mathrm{char}(\varphi_k)>c_0$. This contradicts $b_{2^{c_0}}(k)\to a_{2^{c_0}}$.

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Third case : $\mathrm{char}(\varphi) = c_0 > 0$ and $\mathrm{char}(\varphi_k) < c_0$ for all k.

With a fixed characteristic

Theorem

Let $c_0 \in \mathbb{N}_0$. The set of compact composition operators with characteristic equal to c_0 is arcwise connected.

Preliminaries - Hedenmalm, Lindqvist, Seip

1. $\mathbb{T}^\infty$ can be identified with the dual group of $(\mathbb{Q}_+, \cdot)$ by

$$z \in \mathbb{T}^\infty \mapsto \chi_z,\ \chi_z(2) = z_1,\ \chi_z(3) = z_2,\ \dots,\ \chi_z(p_m) = z_m.$$

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2. For $\chi \in \mathbb{T}^\infty$ and $f = \sum_n a_n n^{-s} \in \mathcal{H}$, define $f_\chi(s) = \sum_n a_n \chi(n) n^{-s}$.

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Proposition

Let $f,\ F \in \mathcal{H}$. TFAE :

- *There exists $(\tau_k) \subset \mathbb{R}$, $f(\cdot + i\tau_k) \to F$ uniformly on compact subsets of $\mathbb{C}_{1/2}$.*
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3. Let $f \in \mathcal{H}$. For almost all $\chi \in \mathbb{T}^\infty$, f_χ converges in $\mathbb{C}_0$, $f_\chi(it) = \lim_{\sigma \to 0} f_\chi(\sigma + it)$ exists for almost all $t \in \mathbb{R}$ and
$$\|f\|^2 = \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(it)|^2 d\mu(t) dm(\chi).$$

Preliminaries - Hedenmalm, Lindqvist, Seip, Gordon

4. Let $f \in \mathcal{H}$, $\varphi = c_0 s + \psi \in \mathcal{G}$. Define $\varphi_\chi = c_0 s + \psi_\chi$. Then

$$(f \circ \varphi)_\chi = f_{\chi^{c_0}} \circ \varphi_\chi.$$

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Fix $f \in \mathcal{H}$ and let us estimate $\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\|$.

$$\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\|^2 \leq \iint |(f\circ\varphi_\lambda)_\chi(it) - (f\circ\varphi_{\lambda'})_\chi(it)|^2 d\mu(t)dm(\chi)$$

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Assume that $\varphi_0(\mathbb{C}_+),\ \varphi_1(\mathbb{C}_+) \subset \mathbb{C}_a$ for some $a > 1/2$. Then $(\varphi_\lambda)_\chi(it) \in \overline{\mathbb{C}_a}$ for all χ and all λ so that

$$|f_{\chi^{c_0}}\circ(\varphi_\lambda)_\chi(it) - f_{\chi^{c_0}}\circ(\varphi_{\lambda'})_\chi(it)|^2 \leq C_a\|f\|^2|(\varphi_\lambda)_\chi(it) - (\varphi_{\lambda'})_\chi(it)|^2.$$

Recall that $\varphi_\lambda = c_0 s + (1-\lambda)\psi_0 + \lambda\psi_1$. Then

$$(\varphi_\lambda)_\chi(it) - (\varphi_{\lambda'})_\chi(it) = (\lambda - \lambda')\big((\psi_0)_\chi(it) - (\psi_1)_\chi(it)\big).$$

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If $|\psi_0|$ and $|\psi_1|$ are bounded, we finally find

$$\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\|^2 \leq M|\lambda-\lambda'|^2 \cdot \|f\|^2.$$

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Until now we have shown that, if $\varphi_0 = c_0 s + \psi_0$ and $\varphi_1 = c_0 s + \psi_1$ are such that

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Consider for $\sigma \in [0,1]$, $\varphi_\sigma = \varphi(\cdot + \sigma)$. Then φ_1 satisfies the above assumptions. Is the map $\sigma \in [0,1] \mapsto C_{\varphi_\sigma}$ continuous?

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The lemma follows from a (standard) compactness argument.

Two general statements

Theorem (Positive characteristic)

Let φ_0 and $\varphi_1 \in \mathcal{G}$ with $\mathrm{char}(\varphi_0) = \mathrm{char}(\varphi_1) =: c_0 \geq 1$ and write them $\varphi_0 = c_0 s + \psi_0$, $\varphi_1 = c_0 s + \psi_1$. Assume moreover that there exists $C > 0$ such that

- $|\varphi_0 - \varphi_1| \leq C \min(\Re e\varphi_0, \Re e\varphi_1)$;
- $|\psi_0|$, $|\psi_1| \leq C$;
- $|\psi_0'|$, $|\psi_1'| \leq C$.

Then C_{φ_0} and C_{φ_1} belong to the same component of $\mathcal{C}(\mathcal{H})$.

Two general statements

Theorem (Positive characteristic)

Let φ_0 *and* $\varphi_1 \in \mathcal{G}$ *with* $\mathrm{char}(\varphi_0) = \mathrm{char}(\varphi_1) =: c_0 \geq 1$ *and write them* $\varphi_0 = c_0 s + \psi_0$, $\varphi_1 = c_0 s + \psi_1$. *Assume moreover that there exists* $C > 0$ *such that*

- $|\varphi_0 - \varphi_1| \leq C \min(\Re e \varphi_0, \Re e \varphi_1)$;
- $|\psi_0|$, $|\psi_1| \leq C$;
- $|\psi_0'|$, $|\psi_1'| \leq C$.

Then C_{φ_0} *and* C_{φ_1} *belong to the same component of* $\mathcal{C}(\mathcal{H})$.

Theorem (Zero characteristic)

Let φ_0 *and* $\varphi_1 \in \mathcal{G}$ *with* $\mathrm{char}(\varphi_0) = \mathrm{char}(\varphi_1) = 0$. *Assume that there exists* $C > 0$ *such that* $|\varphi_0 - \varphi_1| \leq C \min\left(\frac{\Re e \varphi_0 - 1/2}{|1+\varphi_0|^2}, \frac{\Re e \varphi_1 - 1/2}{|1+\varphi_1|^2}\right)$. *Then* C_{φ_0} *and* C_{φ_1} *belong to the same component of* $\mathcal{C}(\mathcal{H})$.

Idea for the proof

Let φ_0, $\varphi_1 \in \mathcal{G}$. As before, define $\varphi_\lambda = (1-\lambda)\varphi_0 + \lambda\varphi_1$. Write

$$\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\|^2 \leq \int\int |f_{\chi^{c_0}}((\varphi_\lambda)_\chi(it)) - f_{\chi^{c_0}}((\varphi_{\lambda'})_\chi(it))|^2 d\mu dm.$$

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Observe that

$$\begin{aligned} f_{\chi^{c_0}}((\varphi_\lambda)_\chi(it)) - f_{\chi^{c_0}}((\varphi_{\lambda'})_\chi(it)) &= \big((\varphi_1)_\chi(it) - (\varphi_0)_\chi(it)\big)\times \\ &\qquad \int_\lambda^{\lambda'} f'_{\chi^{c_0}}((\varphi_r)_\chi(it))dr \end{aligned}$$

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By Jensen's inequality,

$$\begin{aligned} \left\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\right\|^2 &\leq |\lambda' - \lambda|^2 \int_\lambda^{\lambda'} \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |(\varphi_1)_\chi(it) - (\varphi_0)_\chi(it)|^2 \\ &\qquad \times |f'_{\chi^{c_0}}((\varphi_r)_\chi(it))|^2 d\mu(t) dm(\chi) dr. \end{aligned}$$

Therefore, it suffices to show that there exists $C \geq 1$ such that, for all $r \in [0,1]$,

$$\iint |(\varphi_1)_\chi(it) - (\varphi_0)_\chi(it)|^2 \, |(f' \circ \varphi_r)_\chi(it)|^2 d\mu(t) dm(\chi) \leq C \iint |f_\chi(it)|^2 d\mu(t) dm(\chi).$$

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General problem: boundedness of weighted composition operators from $\mathcal{A}$ to $\mathcal{H}$

Let $w : \mathbb{C}_{1/2} \to \mathbb{C}$ be a Dirichlet series, $\varphi \in \mathcal{G}$. When do $wC_\varphi : f \mapsto w \cdot f \circ \varphi$ defines a bounded operator from $\mathcal{A}$ to $\mathcal{H}$?

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Two different proofs:

- For $c_0 = 0$, we reduce to Hardy and Bergman spaces of the unit disc and use Carleson measures.
- For $c_0 \geq 1$, we work directly with Dirichlet series in $\mathcal{A}$ and in $\mathcal{H}$ and use some Nevanlinna counting functions.

Application 1: linear symbols

Let $(q_j)_{j=1,\dots,d}$ be multiplicatively independent positive integers and let $\varphi_0 = c_0 s + c_1 + \sum_{j=1}^d c_{q_j} q_j^{-s} \in \mathcal{G}$, $c_0 \geq 1$.

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This disproves a conjecture of Shapiro and Sundberg in this setting (already disproved on $H^2(\mathbb{D})$ by Bourdon and by Moorhouse and Tonge).

Application 2: coefficients of the Bohr lift

Let $\varphi(s) = c_0 s + \psi(s) \in \mathcal{G}$, $\psi(s) = \sum_{n=1}^N c_n n^{-s}$ be a Dirichlet polynomial symbol with $c_0 \geq 1$. Define the Bohr lift $\mathcal{B}\psi$ of ψ by

$$\mathcal{B}\psi(z) = \sum_{n=p_1^{\alpha_1}\cdots p_d^{\alpha_d}=1}^{N} c_n z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$

Then $\mathcal{B}\psi$ maps $\mathbb{D}^d$ into $\mathbb{C}_+$.

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Then $\mathcal{B}\psi$ maps $\mathbb{D}^d$ into $\mathbb{C}_+$.
Let $\Gamma(\mathcal{B}\psi) = \{z \in \mathbb{T}^d : \Re e(B\psi(z)) = 0\}$.

Definition

Let $z \in \Gamma(\mathcal{B}\psi)$. We say that φ has Dirichlet contact of order n at z if there exists a neighbourhood $\mathcal{U}$ of z in $\mathbb{T}^d$ such that, for all $w \in \mathcal{U}$,

$$\left|\Im m\big(\mathcal{B}_\psi(w) - \mathcal{B}_\psi(z)\big)\right|^{2n} \lesssim \Re e\big(\mathcal{B}_\psi(w)\big).$$

Coefficients of the Bohr lift

Corollary

Let $\varphi_0, \varphi_1 \in \mathcal{G}$ *be Dirichlet polynomial symbols with* $\mathrm{char}(\varphi_0) = \mathrm{char}(\varphi_1) \geq 1$. *Assume that* $\Gamma(\mathcal{B}\psi_0) = \Gamma(\mathcal{B}\psi_1)$ *and that, for all* $z \in \Gamma(\mathcal{B}\psi_0)$, *there exists* $n \in \mathbb{N}$ *such that*

- $\mathcal{B}\psi_0(z) = \mathcal{B}\psi_1(z)$;
- φ_0 *and* φ_1 *have a Dirichlet contact of order* $2n$ *at* z;
- *for* $|\alpha| \leq 2n-1$, $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$.

Then C_{φ_0} *and* C_{φ_1} *belong to the same component of* $\mathcal{C}(\mathcal{H})$.

Open questions and work in progress

1. How to prove that two composition operators do not belong to the same component? In particular, what about $\varphi_0(s) = s + 1 - 2^{-s}$ and $\varphi_1(s) = s + 1 - 3^{-s}$?

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4. Can we use these methods to give conditions implying that $C_{\varphi_0} - C_{\varphi_1}$ is compact?

Theorem

Let φ_0 *and* $\varphi_1 \in \mathcal{G}$ *with* $\mathrm{char}(\varphi_0) = \mathrm{char}(\varphi_1) = 0$. *Assume that*

$$|\varphi_0 - \varphi_1| = o\left(\min\left(\frac{\Re e\varphi_0 - 1/2}{|1+\varphi_0|^2}, \frac{\Re e\varphi_1 - 1/2}{|1+\varphi_1|^2}\right)\right) \text{ as } \Re e(s) \to 0.$$

Then $C_{\varphi_0} - C_{\varphi_1}$ *is compact.*

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Then $C_{\varphi_0} - C_{\varphi_1}$ is compact.

What happens for $c_0 \geq 1$? For instance,

Conjecture

Let $\varphi_0, \varphi_1 \in \mathcal{G}$ be Dirichlet polynomial symbols with $\mathrm{char}(\varphi_0) = \mathrm{char}(\varphi_1) \geq 1$. Assume that $\Gamma(\mathcal{B}\psi_0) = \Gamma(\mathcal{B}\psi_1)$ and that, for all $z \in \Gamma(\mathcal{B}\psi_0)$, there exists $n \in \mathbb{N}$ such that

- $\mathcal{B}\psi_0(z) = \mathcal{B}\psi_1(z)$;
- φ_0 and φ_1 have a Dirichlet contact of order $2n$ at z;
- for $|\alpha| \leq 2n$, $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$.

Then $C_{\varphi_0} - C_{\varphi_1}$ is compact???

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See you soon there!

Thank you!