Random multiplicative functions and a model for the Riemann zeta function

Joint work with Marco Aymone and Jing Zhao

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$$\zeta(\tfrac{1}{2} + it) \sim \sum_{n \leqslant T} \frac{1}{n^{1/2+it}}$$

Key feature: length of sum depends on height T.

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- Example - Moments $\frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} dt$.

$$\frac{1}{T} \int_T^{2T} \left(\frac{m}{n}\right)^{-it} dt = \mathbb{1}_{m=n} + \mathbb{1}_{m \neq n} \cdot \frac{1}{T} \frac{O(1)}{\log(m/n)}$$

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- Replace n^{-it} by random multiplicative $f(n)$ where the $(f(p))_{p \text{ prime}}$ are independent Steinhaus random variables. $f(n) = \prod_{p^\alpha || n} f(p)^\alpha$. E.g. $f(6) = f(2)f(3)$, $f(12) = f(2)^2 f(3)$.

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$$\mathbb{E}[f(m)\overline{f(n)}] = \mathbb{1}_{m=n}.$$

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- We shall consider the following questions/problems:
  (i) Moments: $\mathbb{E}[|M_f(T)|^{2k}]$ for $k \in \mathbb{R}_{>0}$.
  (ii) Distribution of $M_f(T)$.
  (iii) "Independently sampled maxima":
$$\max_{1 \leqslant j \leqslant N} |M_{f_j}(T)|$$
  (iv) "Almost sure" bounds.

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- (Conrey–Gamburd, 2003) For $k \in \mathbb{N}$,

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for some explicit constants $a(k)$ and $\gamma(k)$.

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- Order: (Gerspach, 2019) for $k > 0$

$$\mathbb{E}[|M_f(T)|^{2k}] \asymp_k (\log T)^{k^2}.$$

- Dependence on k is important in determining large values/distribution. For small $k \leqslant 1$, uniform version of Gerspach's result:

$$c(\log T)^{k^2} \leqslant \mathbb{E}[|M_f(T)|^{2k}] \leqslant C \cdot \frac{1}{k^2} \cdot (\log T)^{k^2}.$$

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- (Brevig–Bondarenko–Saksman–Seip–Zhao, 2018)

$$e^{-(2+o(1))k^2 \log k} \leqslant \lim_{T\to\infty} \frac{\mathbb{E}[|M_f(T)|^{2k}]}{(\log T)^{k^2}} \leqslant e^{-(1+o(1))k^2 \log k}$$

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- (H.–Brevig, 2019) Sharp bounds for uniformly large k. Upper bound in the range

$$1 \leqslant k \leqslant c \log T / \log\log T$$

and lower bound for

$$1 \leqslant k \leqslant \sqrt{\log\log T}.$$

Theorem (Aymone, H., Zhao)

We have

$$\mathbb{E}[|M_f(T)|^{2k}] = e^{-k^2 \log k - k^2 \log\log k + O(k^2)} (\log T)^{k^2}$$

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- For $1 \leqslant k \leqslant \log\log T$ we use less wasteful interpolation inequality (Weissler's inequality).

- So assume $k \in \mathbb{N}$. Then by orthogonality

$$\mathbb{E}[|M_f(T)|^{2k}] = \sum_{\substack{n_1 \cdots n_k = n_{k+1} \cdots n_{2k} \\ n_j \leqslant T}} \frac{1}{(n_1 \cdots n_{2k})^{1/2}}.$$

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$$\ll \sum_{\substack{n_1\cdots n_k = n_{k+1}\cdots n_{2k} \\ n_j \in S(Y), n_1 > T}} \frac{1}{(n_1 \cdots n_{2k})^{1/2}}$$

$$\ll \frac{1}{T^\alpha} \sum_{\substack{n_1\cdots n_k = n_{k+1}\cdots n_{2k} \\ n_j \in S(Y)}} \frac{n_1^\alpha}{(n_1 \cdots n_{2k})^{1/2}}.$$

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- Choose $\alpha = 1/\log Y$ then this is tolerable if $Y \leqslant T^{1/ck}$ for some sufficiently large c.

- Main term:

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$$= e^{O(k^2)}\left(\frac{\log T}{k\log k}\right)^{k^2}$$

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- Selberg's Central Limit Theorem: For $V = \Delta\sqrt{\frac{1}{2}\log\log T}$ with fixed Δ we have

$$\frac{1}{T}\mu\left(\left\{t \in [T, 2T] : |\zeta(\tfrac{1}{2}+it)| \geqslant e^V\right\}\right) \sim \frac{1}{\sqrt{2\pi}}\int_\Delta^\infty e^{-x^2/2}dx.$$

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- Note Δ is fixed hence $V \approx \sqrt{\log\log T}$. (Soundararajan, 2009) On RH, the left hand side is

$$\ll \exp\left(- (1 + o(1))\frac{V^2}{\log\log T}\right)$$

in the wider range $V \ll \log_2 T \log_3 T$. Expect distribution to change (mildly) for larger V.

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Theorem (Aymone, H., Zhao)

For $\sqrt{\log_2 T}\log_3 T \leqslant V \leqslant C\log T/\log_2 T$ *we have*

$$\mathbb{P}\Big(|M_f(T)| \geqslant e^V\Big) = \exp\Big(-(1+o(1))\frac{V^2}{\log(\frac{\log T}{V})}\Big).$$

If $V = \Delta\sqrt{\frac{1}{2}\log_2 T}$ *with fixed* Δ *then*

$$\mathbb{P}\Big(|M_f(T)| \geqslant e^V\Big) \gg \int_\Delta^\infty e^{-x^2/2}dx$$

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- E.g. For the upper bound use Chebyshev:

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$$\leqslant \exp(-2kV - k^2\log k + k^2\log_2 T + \cdots)$$

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$$2k\int_{-\infty}^{\infty}\mathbb{P}(|M| \geqslant e^u)e^{2ku}du \overset{\exists k=k(V)}{\approx} 2k\int_{V(1-\epsilon)}^{V(1+\epsilon)}\mathbb{P}(|M| \geqslant e^u)e^{2ku}du$$

- Gaussian bound when $V = \Delta\sqrt{\frac{1}{2}\log\log T}$ is out of range of moments. We use developments of Harper on Helson's conjecture to relate $M_f(T)$ to the Euler product $\approx \exp(\sum_{p \leqslant T} f(p)p^{-1/2})$.

Applications to zeta

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- Therefore, sampling zeta at $T\log T$ independent points should pick up the maximum $\max_{t\in[T,2T]} |\zeta(\frac{1}{2}+it)|$.
- This gives us a model for the max of zeta:

$$\max_{1\leqslant j\leqslant T\log T} |M_{f_j}(T)|$$

where the f_j are independently sampled.

A model for the max of zeta

Theorem (Aymone, H., Zhao)

Let $c \geqslant 1/\sqrt{2}$. Then

$$\mathbb{P}\left(\max_{1 \leqslant j \leqslant T \log T} |M_{f_j}(T)| \leqslant \exp(c\sqrt{\log T \log\log T})\right) = 1 - o(1)$$

as $T \to \infty$. If $c < 1/\sqrt{2}$ the probability is $o_{T\to\infty}(1)$.

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- Compare with conjecture of (Farmer, Gonek, Hughes, 2005):

$$\max_{t\in[T,2T]} |\zeta(\tfrac{1}{2} + it)| = \exp\left((1 + o(1))\sqrt{\tfrac{1}{2}\log T \log\log T}\right).$$

- Proof: Let $V = c\sqrt{\log T \log_2 T}$. By independence of the f_j, the probability is

$$\mathbb{P}\big(|M_f(T)| \leqslant e^V\big)^{T \log T} = \big(1 - \mathbb{P}(|M_f(T)| \geqslant e^V)\big)^{T \log T}$$

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$$= \exp\Big(- T(\log T) \cdot \mathbb{P}(|M_f(T)| \geqslant e^V) + \cdots \Big).$$

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$$= \exp\Big(- T(\log T) \cdot \mathbb{P}(|M_f(T)| \geqslant e^V) + \cdots \Big).$$

Now input tail bounds for the probability. □

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Model for $\sum_{n \leqslant x} \mu(n)$.

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- (Wintner, 1944)

$$S_f(x) \ll x^{1/2+\epsilon} \qquad \text{a.s.}$$

Improvements by Erdös, Halasz, and most recently (Lau–Tenenbaum–Wu, Basquin, 2012):

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- (Harper, 2020)

$$S_f(x) \neq O(x^{1/2} (\log \log x)^{1/4-\epsilon}) \qquad \text{a.s.}$$

- Transferring to our case:

$$M_f(T) \ll \log T (\log\log T)^{1/4+\epsilon}, \qquad \text{a.s.}$$

and

$$M_f(T) \neq O(1) \qquad \text{a.s.}$$

Theorem (Aymone, H., Zhao)

We have

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- Upper bound – squareroot cancellation: For independent (X_n) expect

$$\sum X_n \overset{\approx}{\ll} \sqrt{\mathbb{E}|\sum X_n|^2} \qquad \text{a.s.}$$

and

$$\mathbb{E}[|M_f(T)|^2] = \sum_{n\leqslant T} \frac{1}{n} \approx \log T.$$

- Lower bound – Connection with Euler product:

$$M_f(T) \overset{?}{\approx} \prod_{p \leqslant T} \left(1 - \frac{f(p)}{\sqrt{p}}\right)^{-1} \approx \exp\left(\sum_{p \leqslant T} \frac{f(p)}{\sqrt{p}}\right)$$

and

$$\sqrt{\mathbb{E}\left[\left|\sum_{p \leqslant T} \frac{f(p)}{\sqrt{p}}\right|^2\right]} = \sqrt{\sum_{p \leqslant T} \frac{1}{p}} = \sqrt{\log \log T}$$

- Proof ideas (Upper bound) – Main tool is Borel–Cantelli:

Theorem

Let E_n be some events. If $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$ then

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- Example: Take r.v's (X_n) with $\mathbb{P}(X_n \geqslant V) = e^{-V}$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(X_n \geqslant (1+\epsilon)\log n\Big) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty.$$

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Hence, $X_n \ll \log n \quad$ a.s.

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- Made possible since $M_f(T)$ is slowly varying:

Lemma

Let $T_j = e^{j^4}$ with $j \geqslant 1$. Then

$$\max_{T_{j-1} < T \leqslant T_j} |M_f(T) - M_f(T_{j-1})| \ll (\log T_j)^{1/2+\epsilon} \qquad a.s.$$

Maximal inequalities

- We prove this lemma in the same way using Borel-Cantelli. Requires summable bounds on

$$(*) \qquad \mathbb{P}\Bigg(\max_{T_{j-1} < T \leqslant T_j} \Bigg| \sum_{T_{j-1} < n \leqslant T} \frac{f(n)}{\sqrt{n}} \Bigg| \geqslant (\log T_j)^{1/2+\epsilon} \Bigg).$$

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- If $f(n)$ were independent we could use e.g. Kolmogorov's maximal inequality: X_i independent, $S_j = \sum_{i \leqslant j} X_i$, then

$$\mathbb{P}\left(\max_{1 \leqslant j \leqslant n} |S_j| \geqslant V \right) \leqslant \frac{\mathbb{E}[|S_n|^2]}{V^2}.$$

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- Naively applying this:

$$(*) \leqslant \frac{1}{(\log T_j)^{1+\epsilon}} \sum_{T_{j-1} < n \leqslant T_j} \frac{1}{n} \ll \frac{\log(T_j/T_{j-1})}{(\log T_j)^{1+\epsilon}} \ll \frac{1}{j^{1+\epsilon}}$$

recalling $T_j = e^{j^4}$.

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Lemma

Let X_i be random variables, $S_j = \sum_{i \leqslant j} X_i$ and suppose there exists $\alpha > 1$, $\beta > 0$ and coefficients u_i such that

$$\mathbb{P}(|S_l - S_k| \geqslant V) \leqslant \frac{1}{V^\beta} \Big(\sum_{k \leqslant i \leqslant l} u_i \Big)^\alpha$$

for all $0 \leqslant k \leqslant l \leqslant n$. Then

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- $$\mathbb{E}[|M_f(l) - M_f(k)|^2] = \sum_{k<n\leqslant l} \frac{1}{n}, \mathbb{E}[|M_f(l) - M_f(k)|^4] \leqslant \Big(\sum_{k<n\leqslant l} \frac{d(n)}{n}\Big)^2$$

- Interpolate:

$$\mathbb{E}[|M_f(l) - M_f(k)|^{2+\delta}] \lessapprox \Bigg(\sum_{k<n\leqslant l} \frac{(\log n)^{\delta}}{n} \Bigg)^{1+\delta}.$$

This gives, by Chebyshev,

$$\mathbb{P}(|M_f(l) - M_f(k)| \geqslant V) \leqslant \frac{1}{V^{2+\delta}} \Bigg(\sum_{k<n\leqslant l} \frac{(\log n)^{\delta}}{n} \Bigg)^{1+\delta}$$

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- $$(*) \leqslant \frac{1}{(\log T_j)^{1+\epsilon}} \left(\sum_{T_{j-1}<n\leqslant T_j} \frac{(\log n)^\delta}{n} \right)^{1+\delta} \ll \frac{1}{j^{1+\epsilon-2\delta}} \quad \square$$

Lower bounds

- Want to show

$$\limsup_{T\to\infty} \frac{|M_f(T)|}{\exp(\Delta\sqrt{\log\log T})} = \infty \qquad \text{a.s.}$$

for any $\Delta > 0$.

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"If $\mathcal{A}$ is a tail event then $\mathbb{P}(\mathcal{A}) = 0$ or 1".

Lemma

Let $\lambda(T) \to \infty$. *The event*

$$\mathcal{A}_\lambda = \Big\{ |M_f(T)| \geq \exp(\lambda(T)) \textit{ for infinitely many integers } T > 0 \Big\}$$

is a tail event with respect to the $(f(p))_{p \text{ prime}}$.

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- Taking $\lambda(T) = \Delta\sqrt{\log\log T}$ we have

$$\mathbb{P}(|M_f(T)| \geqslant \exp(\lambda(T))) \gg \int_\Delta^\infty e^{-x^2/2} dx > 0,$$

and hence, since

$$\mathcal{A}_\lambda = \bigcap_{n=1}^{\infty} \bigcup_{T=n}^{\infty} [|M_f(T)| \geq \exp(\Delta\sqrt{\log\log T})],$$

we have

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- $\mathbb{P}(\mathcal{A}_\lambda) = 1$ by the zero-one law.

- Proof sketch of Lemma: Let $f_{\geqslant y}(n)$ be $f(n)$ supported on primes $p \geqslant y$. Then

$$M_{f_{\geqslant y}}(T) = \sum_{n \leqslant T} \frac{\mu(n) f_{\leqslant y}(n)}{\sqrt{n}} M_f(T/n)$$

"since" $\prod_{p \geqslant y} = \prod_p / \prod_{p \leqslant y}$.

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"since" $\prod_{p \geqslant y} = \prod_p / \prod_{p \leqslant y}$. Thus, if

$$M_{f_{\geqslant y}}(T) \geqslant e^{\lambda(T)}$$

then

$$M_f(T/n) \geqslant 2^{-\pi(y)} e^{\lambda(T)}$$

for infinitely many n. Reverse inclusion similar.

- Thanks!