Restriction Theorems for Multiplicative Hankel Operators

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Dirichlet Series and Operator Theory Wokshop
Oslo, June 17, 2021

Multiplicative Hankel matrices

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$$M(\alpha) \simeq H_\infty(\varphi), \quad \varphi(s) = \sum_{n\geqslant 1} \alpha(n) n^{-s}.$$

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$\mathbf{M}(\mathbf{a})$ is unitarily equivalent the (classical) Hankel operator $H(\psi)$ (on $H^2(\mathbb{R})$),

$$\psi(\xi) = \hat{\mathbf{a}}(\xi) = \int_1^\infty \mathbf{a}(t)t^{-1/2+i\xi}\,dt.$$

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Action on symbols: Formally the (analytic) symbol for $M(\alpha)$ is

$$\varphi(s) = \int_{-\infty}^{\infty} \zeta(s + 1/2 + i\xi)\widehat{\mathbf{a}}(\xi)\, d\xi.$$

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$$||M(\alpha)||_{\mathbf{S}_p} \leqslant C_p ||\mathbf{M}(\mathbf{a})||_{\mathbf{S}_p}.$$

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Moreover, if $\mathbf{M}(\mathbf{a}) \in \mathbf{S}_{p,\infty}$ *then* $M(\alpha) \in \mathbf{S}_{p,\infty}$ *and*

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- Note that $\mathbf{M}(\mathbf{a}) \in \mathbf{S}_1 \Rightarrow \mathbf{a}(t)$ is continuous for $t > 1$.
- This results fails for any $p > 1$.

Lemma

Suppose $\operatorname{supp}(\mathbf{a}) \subseteq [1, e^N]$ *and* $\hat{\mathbf{a}} \in L^p(\mathbb{R})$, $0 < p \leqslant 1$. *Then*

$$\|M(\alpha)\|_{\mathbf{S}_p}^p \leqslant C_p N \|\hat{\mathbf{a}}\|_{L^p(\mathbb{R})}^p.$$

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Proof based on series representation for $\mathbf{a}(t)$:

$$\mathbf{a}(t) = \frac{1}{N} \sum_{m=-\infty}^{\infty} \hat{\mathbf{a}}(m/N) t^{-\frac{1}{2} + i\frac{m}{N}}, \quad 1 < t < e^N,$$

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Hence

$$\|M(\alpha)\|_{\mathbf{S}_p}^p \leqslant \frac{1}{N^p} \sum_{m=-\infty}^{\infty} |\hat{\mathbf{a}}(m/N)|^p (N+1)^p \leqslant C_p N \|\hat{\mathbf{a}}\|_{L^p(\mathbb{R})}^p.$$

Sketch proof of theorem

Take a "dyadic" decomposition of $\mathbf{a}$:

$$\mathbf{a} = \sum_{n=-\infty}^{\infty} \mathbf{a}_n, \quad \operatorname{supp}(\mathbf{a}_n) \subseteq \left[\exp\left(2^{n-1}\right), \exp\left(2^{n+1}\right)\right].$$

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Use real interpolation to get weak-$\mathbf{S}_p$ estimates. □

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Suppose $\mathbf{M}(\mathbf{a})$ is non-negative. Then the following hold:

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Question: Does this hold for all $0 < p < \infty$?

Spectral properties

Theorem

Let $w \geqslant 0$ *be a bounded function on* $\mathbb{R}_+$ *with bounded support. Let*

$$\mathbf{a}(t) = \int_0^\infty t^{-\frac{1}{2}-\lambda} w(\lambda) d\lambda, \quad t > 1.$$

Then $M(\alpha) \approx \mathbf{M}(\mathbf{a}) + A$ *with* $A \in \cap_{p>0} \mathbf{S}_p$.

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$$s_n=C(\gamma)n^{-\gamma}+o(n^{-\gamma}), \quad n\to\infty.$$

Similar results for integral Hankel operators

Modification of Multiplicative Hilbert matrix

Multiplicative Hilbert matrix: $M(\alpha), \quad \alpha(j) = 1/\sqrt{j}\log j$

(Brevig-Perfekt-Seip-Siskakis-Vukotić 2016, Perfekt-Pushnitski 2018)

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Theorem

Let $\gamma > 0$. Suppose that for all sufficiently large j,

$$\alpha(j) = 1/(\sqrt{j}\log j(\log\log j)^{\gamma}).$$

Then $M(\alpha)$ is compact and the singular values satisfy

$$s_n(M(a)) = C(\gamma)n^{-\gamma} + o(n^{-\gamma}), \quad n \to \infty.$$

Modification of Multiplicative Hilbert matrix

Multiplicative Hilbert matrix: $M(\alpha)$, $\quad \alpha(j) = 1/\sqrt{j}\log j$

(Brevig-Perfekt-Seip-Siskakis-Vukotić 2016, Perfekt-Pushnitski 2018)

Theorem

Let $\gamma > 0$. Suppose that for all sufficiently large j,

$$\alpha(j) = 1/(\sqrt{j}\log j(\log\log j)^{\gamma}).$$

Then $M(\alpha)$ is compact and the singular values satisfy

$$s_n(M(a)) = C(\gamma)n^{-\gamma} + o(n^{-\gamma}), \quad n \to \infty.$$

Question: Does a version of Widom's theorem hold for Helson matrices?

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Result follows from restricting to integers and applying standard spectral perturbation theory.

Thank you!