On an operator theoretic proof of the Prime Number Theorem

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Overview

- Background
  - Some facts on the Riemann zeta function.
  - Tauberian approach.
- Main part
  - Operator theoretic tauberian theorem (w/proof)
  - Deduction of the prime number theorem
- Remarks
  - Connection to function spaces of Dirichlet series.
  - Possible connection to Quantum Harmonic Analysis.

Background

Hadamard, de la Vallé Poussin (1896)

Let $\pi_{\mathbb{P}}$ be the counting function for the prime numbers. Then

$$\pi_{\mathbb{P}}(x) \sim \frac{x}{\log x}.$$

A selection of the history of the PNT:

Euler (1737)

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Chebyshev (1851-52)

$$0.89 \cdot \frac{x}{\log x} \leq \pi_{\mathbb{P}}(x) \leq 1.11 \cdot \frac{x}{\log x}, \qquad x \to \infty$$

Riemann (1859)

$$\zeta(s) = \frac{1}{s-1} + \phi(s), \qquad \phi \text{ entire}$$

von Mangoldt (1894)

The statement "$\zeta(s) = 0 \implies \operatorname{Re} s < 1$" implies the PNT.

Hadamard, de la Vallé Poussin (1896)

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- 1907: E. Landau came up with the tauberian approach.

Landau (1907)

Suppose:

- $a_n \geq 0$ for all $n \in \mathbb{N}$.
- $G(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges on $\mathrm{Re}(s) > 1$.
- $G(s) - A/(s-1)$ has analytic continuation to $\mathrm{Re}(s) = 1$ for $A \geq 0$.
- $G(s) = \mathcal{O}\left(|s|^{\alpha}\right)$ for some $\alpha > 0$ on $\mathrm{Re}(s) = 1$.

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = A.$$

- Proof involves delicate analysis.
- Can be used to prove PNT.

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Landau-Ikehara (1931)

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- 1907: E. Landau came up with the tauberian approach.

Landau-Ikehara-Wiener (1932)

Suppose:

- $S(u)$ a non-decreasing function on $[0, \infty)$.
- $G(s) = \mathcal{L}\{dS(u)\}(s)$ converges on $\mathrm{Re}(s) > 1$.
- $G(s) - A/(s-1)$ has analytic continuation to $\mathrm{Re}(s) = 1$ for $A \geq 0$.

Then

$$\lim_{u \to \infty} \frac{S(u)}{\mathrm{e}^u} = A.$$

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Landau-Ikehara-Wiener-Korevaar (2005)

Suppose:

- $S(u)$ a non-decreasing function on $[0, \infty)$,
- $G(s) = \mathcal{L}\{S(u)\}(s)$ converges on $\mathrm{Re}(s) > 1$.

Then

- $G(s) - A/(s-1)$ extends to a pseudo-function on all finite intervals on $\mathrm{Re}(s) = 1$

if and only if

$$\lim_{u\to\infty} \frac{S(u)}{e^u} = A.$$

- Proof involves delicate analysis.
- Can be used to prove PNT.

Operator theoretic approach

Definition

Suppose

- $S(x)$ defined on $[0, \infty)$.
- $G(s) = \mathcal{L}\{S(e^u)\}(s)$ converges on $\mathrm{Re}(s) > 1$.

For $T > 0$, we define the following operators on $L^2(-T, T)$:

$$W_{S,T,\epsilon} f = \frac{1}{\pi} \int_{-T}^{T} f(\tau) \,\mathrm{Re}\, G\Big(1 + \epsilon + \mathrm{i}(t - \tau)\Big) \mathrm{d}\tau \qquad (\epsilon > 0)$$

$$W_{S,T} f = \lim_{\epsilon \to 0^+} W_{S,T,\epsilon} f.$$

Theorem

Suppose:

- $S(x)$ a non-decreasing function on $[0, \infty)$.
- $G(s) = \mathcal{L}\{S(e^u)\}(s)$ converges on $\mathrm{Re}(s) > 1$.

Then

- $W_{S,T} = A \cdot Id_{L^2(-T,T)} + \Phi_{S,T}$ for $A > 0$ and $\Phi_{S,T}$ compact for all $T > 0$ large enough

if and only if

$$\lim_{x \to \infty} \frac{S(u)}{e^u} = A.$$

- Proof.

Remarks

Connection to function spaces of Dirichlet series

Definition

$$\mathcal{H}_w = \left\{ f(s) = \sum_{n\in\mathbb{N}} a_n n^{-s} : \sum_{n\in\mathbb{N}} |a_n|^2/w_n < \infty \right\}.$$

- Suppose that all functions in $\mathcal{H}_w$ are analytic on $\operatorname{Re} s > 1/2$ and on a dense subset of $\mathcal{H}_w \oplus \overline{\mathcal{H}_w}$ define the operator

$$E_T f(t) = f(1/2 + it)$$

Theorem (O. 2012)

(i) E_T is bounded from $\mathcal{H}_w$ to $L^2(-T,T)$ for all $T > 0$ if and only if

$$\sum_{n\le x} w_n \lesssim x \quad \text{as} \quad x \to \infty.$$

(ii) Such E_T are onto $L^2(-T,T)$ for $T > 0$ sufficiently large if and only if

$$\sum_{n\le x} w_n \gtrsim x \quad \text{as} \quad x \to \infty.$$

- The connection to the current discussion is the following:

$$W_{T,S} = E_T^* E_T + \Phi_T,$$

where Φ_T is compact and $S(x) = \sum_{n\le x} w_n$.

Theorem (O. 2012)

(i) $E_T^* E_T$ bounded on $L^2(-T,T)$ for all $T > 0 \iff \sum_{n\le x} w_n \lesssim x$

(ii) $E_T^* E_T$ bounded below on $L^2(-T,T)$ for $T > 0$ large $\iff \sum_{n\le x} w_n \gtrsim x$

REMARKS

Connection to function spaces of Dirichlet series

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$$\mathcal{H}_w = \Big\{ f(s) = \sum_{n\in\mathbb{N}} a_n n^{-s} : \sum_{n\in\mathbb{N}} |a_n|^2/w_n < \infty \Big\}.$$

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- The connection to the current discussion is the following:
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where Φ_T is compact and $S(u) = \sum_{n\le \mathrm{e}^u} w_n$.

Theorem

(i) $W_{S,T}$ bounded on $L^2(-T,T)$ for all $T>0$ $\iff$ $\sup \frac{S(\mathrm{e}^u)}{\mathrm{e}^u} < \infty$

(ii) $W_{S,T}$ bounded below on $L^2(-T,T)$ for $T>0$ large $\iff$ $\inf \frac{S(\mathrm{e}^u)}{\mathrm{e}^u} > 0.$

- Also exists version where L^2 is replaced by Sobolev spaces.

REMARKS

Connections to Quantum Harmonic Analysis:

- Ikehara's proof uses Wiener's tauberian theorem.

Theorem (Wiener, 1932)

Suppose $f \in L^\infty(\mathbb{R}^d)$. Then

(i) $\exists h \in L^1(\mathbb{R}^d)$ such that $\hat{h} \neq 0$ and $(h * f)(x) = A \int_{\mathbb{R}} h(y) \mathrm{d}y + o(1)$.

implies

(ii) $\forall h \in L^1(\mathbb{R}^d)$ we have $(h * f)(x) = A \int_{\mathbb{R}} h(y) \mathrm{d}y + o(1)$.

- In 1984, Werner established operator analogue in the context of "Quantum Harmonic Analysis".
- There are several versions, here is one:

Theorem (Werner, 1984)

Suppose $f \in L^\infty(\mathbb{R}^{2d})$. Then

$$\exists S \in \mathcal{S}^1 \text{ such that } \mathcal{F}_W(S) \neq 0 \text{ and } f \star S = A \cdot tr(S) \cdot Id + K,$$

implies

$$\forall S \in \mathcal{S}^1 \text{ we have } f \star S = A \cdot tr(S) \cdot Id + K_S,$$

where K and K_S are compact operators on $L^2(\mathbb{R}^{2d})$.

- Joint with Luef and Skrettingland, an investigation into possible connections are under investigation.

Thank you for your attention!