

$$\text{Euler} \Rightarrow \log S(s) = \sum \log \frac{1}{1-p^{-s}}$$

$$= \underbrace{\sum p^{-s}}_{J_p(s)} + \underbrace{\sum G(p^{2j})}_{\varphi}$$

and. on  
 $\operatorname{Re} s > \frac{1}{2}$ .

Riemann's formula.

$$\begin{aligned} S(s) &= \sum n^{-s} = \int_1^\infty x^{-s} d\pi(x) \quad \text{sh } (\pi(e^u))(s) \\ &= x^{-s} \pi(x) \Big|_1^\infty + s \int_1^\infty x^{-s-1} \underbrace{\pi(x) dx}_{\approx x} \\ &= 0 + s \underbrace{\int_1^\infty x^{-s} dx}_{\frac{s}{s-1}} + s \int_1^\infty x^{-s-1} \underbrace{(\pi-x) dx}_{\approx 1} \\ &\qquad\qquad\qquad \text{and } \operatorname{Re} s > 0 \end{aligned}$$

$$\Rightarrow \frac{1}{s} S(s) = h(\pi(e^u)).$$

$$\underline{\text{Mengen: }} S(x) = \sum_{n \leq x} a_n^{-s} = \frac{1}{s} h(S(e^u))(s).$$

Proof that  $\operatorname{Re} s = 1 \not\Rightarrow \zeta(s) \neq 0$ .

$$\text{Assume } \zeta(s) = 0 \Rightarrow \operatorname{Re} \log \zeta(\sigma + it) \xrightarrow{\sigma \rightarrow 1^+} 0$$

$$\begin{aligned} & \Rightarrow \operatorname{Re} \zeta_P(\sigma + it) \rightarrow -\infty \\ & \left( \operatorname{Re} \zeta_P(\sigma + it) \right)^2 \\ &= \left( \sum \frac{\cos(it_0 \log p)}{p^{\sigma/2}} \frac{1}{p^{it_0}} \right)^2 \stackrel{(s \rightarrow s)}{\Rightarrow} \operatorname{Re} \zeta_P(\sigma + 2it) \rightarrow +\infty \\ & \leq \left( \sum \frac{\cos(it_0 \log p)}{p^{\sigma}} \right) \underbrace{\left( \sum \frac{1}{p^{\sigma}} \right)}_{\zeta_P(\sigma)} \Rightarrow \operatorname{Re} \log \zeta(\sigma + 2it) \rightarrow \infty \\ & \boxed{\frac{1}{2}} \boxed{\zeta_P(\sigma + 2it) + \zeta_P(\sigma)} \Rightarrow \zeta(s) \text{ has pole at } s = 1 + 2it. \end{aligned}$$

L-I  $\Rightarrow$  PNT

$$\mathcal{I}_{\text{IP}}(s) = \sum_{p} p^{-s} = \log \frac{1}{s-1} + \varphi$$

||

$$-G(s) = \mathcal{I}_{\text{IP}}^1(s) = -\sum \log p \cdot p^{-s} = -\frac{1}{s-1} + \varphi'$$

||

$$\frac{1}{n} \sum_{p \leq n} \log p \longrightarrow 1.$$

↑ d. avg. (CLT)

Part.

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Ex:  $S(u) = \pi_{\text{HP}}(e^u)$

$$G(s) = \frac{1}{s} S(s) = \frac{1}{s-1} + \varphi(s)$$

$$W_{S,T} f = f + \overline{\int f} e^{t \varphi}$$

Eq. l. op.

Then  $\Rightarrow$  PNT

$$\frac{1}{s} \mathcal{I}_{\text{IP}}(s) = h(\pi_{\text{IP}}^{(r)}(s))$$

$$\begin{aligned}
 &= \int_s^\infty \pi_P(\epsilon^u) \bar{e}^{su} du \\
 \Downarrow & \\
 -G(s) = (\frac{1}{s} \zeta_P(s))^{-1} &= -\int_s^\infty \pi_P(\epsilon^u) u \bar{e}^{su} du \\
 &\quad \Downarrow \\
 -\frac{1}{s-1} + \underbrace{\psi(s)}_{\text{ext. to } L_{loc}} &= -L(\pi_P(\epsilon^u)u)(s) \\
 &\quad \text{on Res} \downarrow \\
 W_{S,T} &= 1 + \bigoplus_{\epsilon \in \mathbb{C}P^1} \\
 &\quad \Downarrow \\
 \pi_P(x) \frac{\log x}{x} &= \frac{\pi_P(\epsilon^u)u}{e^u} \rightarrow 1. \quad \square
 \end{aligned}$$

Proof "↑":

$$\begin{aligned}
 \text{Suppose } \frac{s(u)}{e^u} &\rightarrow A. & F(s(\epsilon^u) e^{-u})^{(t-\tau)} \\
 W_{S,T,\epsilon} f &= \frac{1}{2\pi i} \int_{-T}^T f(\tau) \overbrace{\int_{\mathbb{C}P^1} L(s(\epsilon^u)) (1+\epsilon+i(t-\tau)) dt}^+ d\tau
 \end{aligned}$$

$$\stackrel{P1.}{=} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(u) S(e^{iu}) e^{-(1+\varepsilon)|u|} e^{itu} du$$

$$\xrightarrow{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(u) S(e^{iu}) \underbrace{e^{-|u|}}_{\sim} e^{itu} du$$

↓                      bdd

$$\Phi_{S,T} f = \psi_{S,T} f - Af$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(u) \left( \underbrace{\frac{S(e^{iu})}{e^{iu}} - A}_{\sim} \right) e^{iu} du$$

↓                      → 0

comp. on  $L^2(-T, T)$ .

Proof of "↓":

First, we show  $\psi_{S,T}$  bdd  $\Rightarrow \frac{S(e^{iu})}{e^{iu}}$  bdd.

Let  $\{e_n\}$  denote std. basis of  $L^2(-T, T)$

↓

$$\langle \psi_{S,T} e_n, e_n \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{S(e^{iu})}{e^{iu}} |\widehat{e_n}(u)|^2 du.$$

Suppose  $\frac{S(\tau^{u_n})}{e^{u_n}}$  unbdd.

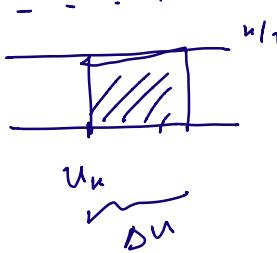


$\exists u_n \rightarrow \infty$  s.t.  $\frac{S(\tau^{u_n})}{e^{u_n}} \rightarrow k$ .



S non-decr.

$$\frac{S(e^{u_n + \Delta u})}{e^{u_n + \Delta u}} \geq \frac{S(e^{u_n})}{e^{u_n}} \cdot \frac{1}{e^{\Delta u}} \geq \frac{1}{e^{\Delta u}}$$



$$\langle w_{s,T, \Delta u}, \epsilon_n \rangle \rightarrow \infty.$$

Next, suppose  $\Phi_{s,T} = \psi_{s,T} - A \text{Id}$  cpt.

Suppose  $\frac{S(\tau^u)}{e^u} \not\rightarrow A$ .



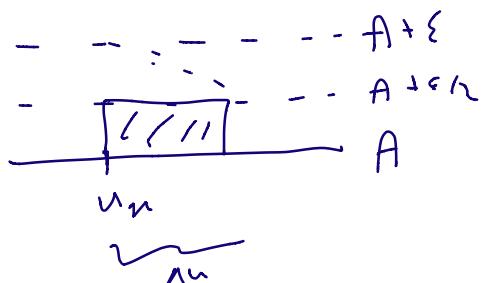
S non-decr.

"WLOG":  $\exists u_n$  s.t.  $\frac{S(e^{u_n})}{e^{u_n}} - A \geq \epsilon$



$$\frac{S(e^{u_n + \Delta u})}{e^{u_n + \Delta u}} - A \geq \frac{S(e^{u_n})}{e^{u_n}} \cdot \frac{1}{e^{\Delta u}} - A$$

$$\geq (A + \varepsilon) \frac{1}{\ell^{\alpha_n}} - A \geq \frac{\varepsilon}{2}$$

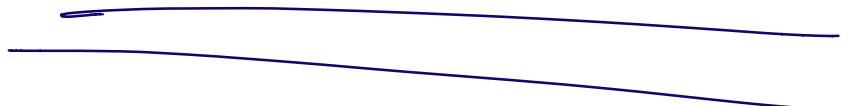


$$\frac{A + \varepsilon}{A + \varepsilon/2} \geq \ell^{\alpha_n}$$

$$\langle \phi_{s,T}^{e_n, e_n} \rangle = \int_{\mathbb{R}} \left( \frac{s(\varepsilon_n)}{\varepsilon_n} - A \right) |Q_n(u)|^2$$

$T$  large enough

$$\langle \psi_{s,T}^{e_n, e_n} \rangle \rightarrow 0. \quad \#.$$



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