Idempotent Fourier multipliers acting contractively on H^p spaces

Joaquim Ortega Cerdà

Universitat de Barcelona

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Joint work with Ole Fredrik Brevig and Kristian Seip

Our problem

- Goal: Describe idempotent Fourier multipliers that act contractively on $H^p(\mathbb{T}^d)$ for $1 \le p \le \infty$, $p \ne 2$, and $d \ge 1$.

First an easier problem:

- Describe idempotent Fourier multipliers that act contractively on $L^p(\mathbb{T}^d)$.

Some notation and terminology: We represent functions f in $L^p(\mathbb{T}^d)$ by their Fourier series $f(z) \sim \sum_{\alpha \in \mathbb{Z}^d} \widehat{f}(\alpha)\, z^\alpha$, where

$$\widehat{f}(\alpha) := \int_{\mathbb{T}^d} f(z)\, \overline{z^\alpha}\, dm_d(z)$$

and m_d denotes the Haar measure of the d-dimensional torus $\mathbb{T}^d$. For Λ a non-empty subset of $\mathbb{Z}^d$,

$$P_\Lambda f(z) := \sum_{\alpha \in \Lambda} \widehat{f}(\alpha) z^\alpha.$$

We say that Λ is a *contractive projection set for* $L^p(\mathbb{T}^d)$ when P_Λ extends to a contraction on $L^p(\mathbb{T}^d)$. A subset Λ of $\mathbb{Z}^d$ is a *coset* in $\mathbb{Z}^d$ if Λ is equal to the coset of a subgroup of $(\mathbb{Z}^d, +)$.

Contractive projection sets for $L^p(\mathbb{T}^d)$

Theorem (After Andô (1965) and Rudin (1962))

Let d be a non-negative integer and fix $1 \leq p \leq \infty$, $p \neq 2$. A subset Λ of $\mathbb{Z}^d$ is a contractive projection set for $L^p(\mathbb{T}^d)$ if and only if Λ is a coset in $\mathbb{Z}^d$.

The proof of the L^p theorem—necessity

Lemma (Linear reflection)

Fix $1 \le p \le \infty$, $p \ne 2$, *and set* $c_p := 2/p - 1$. *Then*

$$\|c_p \varepsilon \overline{z} + 1 + \varepsilon z\|_p < \|1 + \varepsilon z\|_p$$

for every sufficiently small $\varepsilon > 0$.

Lemma (Triangular reflection)

Fix $1 \le p < \infty$, $p \ne 2$, *and set* $c_p := 1 - p/2$.

$$\|1 + \varepsilon(z_1 + z_2) + c_p\, \varepsilon^2 z_1 z_2\|_p < \|1 + \varepsilon(z_1 + z_2)\|_p$$

for every sufficiently small $\varepsilon > 0$.

Think of the z above as $z^{\alpha-\beta}$ with α, β in Γ. Then all linear and triangular reflections constitute the 1-extension of Γ.

Illustration of extension by linear and triangular reflections

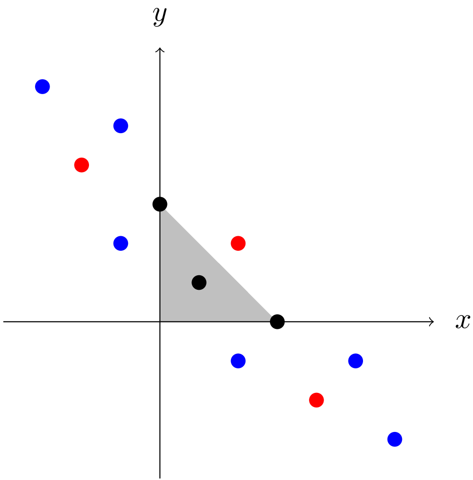

Figure: The points λ obtained by linear and triangular reflection starting from the set $\Gamma = \{(3,0,0),(0,3,0),(1,1,1)\}$, in the plane $z = 3 - x - y$. The shaded triangle represents the intersection of this plane and the narrow cone.

Frequently encountered examples

- F. Wiener's inequality, appearing already in classical work of Bohr's (1914): The case $d = 1$ of the above theorem.

$$Pf(z) = \sum_{k\in\mathbb{Z}} \hat{f}(kn)z^{kn} = \frac{1}{n}\sum_{j=0}^{n-1} f(zw^j)$$

  where w is primitive n'th root of unity.

- The restriction to the m-homogeneous terms of a power series in d variables.

$$Pf(z) = \int_{\mathbb{T}} f(z_1\zeta, z_2\zeta, \ldots, z_d\zeta)\,\overline{\zeta^m}\,dm_1(\zeta)$$

Contractive projection sets for $H^p(\mathbb{T}^d)$

- $H^p(\mathbb{T}^d)$ is the subspace of $L^p(\mathbb{T}^d)$ comprised of functions f with $\widehat{f}(\alpha) = 0$ for every α in $\mathbb{Z}^d \setminus \mathbb{N}_0^d$, where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.
- A subset Λ of $\mathbb{N}_0^d$ is a *contractive projection set for* $H^p(\mathbb{T}^d)$ if P_Λ extends to a contraction on $H^p(\mathbb{T}^d)$.

If Λ is a coset in $\mathbb{Z}^d$, then $\Lambda \cap \mathbb{N}_0^d$ is a contractive projection set for $H^p(\mathbb{T}^d)$.

Question: Are there other contractive projection sets for $H^p(\mathbb{T}^d)$?

The dimension of the affine span of Λ, $\dim(\Lambda)$ plays a nontrivial role in this problem.

Definition

Suppose that $1 \leq k \leq d$. We say that $H^p(\mathbb{T}^d)$ enjoys the *contractive restriction property of dimension* k if every k-dimensional contractive projection set for $H^p(\mathbb{T}^d)$ is of the form $\Lambda \cap \mathbb{N}_0^d$ with Λ a coset in $\mathbb{Z}^d$.

Main theorem (from low to high dimensions)

Theorem

Suppose that $1 \le p \le \infty$.

(a) *If $d = 2$ or $k = 1$, then $H^p(\mathbb{T}^d)$ enjoys the contractive restriction property of dimension k if and only if $p \neq 2$.*

(b) *If either $d = k = 3$ or $d \ge 3$ and $k = 2$, then $H^p(\mathbb{T}^d)$ enjoys the contractive restriction property of dimension k if and only if $p \neq 2, 4$.*

(c) *If $d \ge 4$ and $k \ge 3$, then $H^p(\mathbb{T}^d)$ enjoys the contractive restriction property of dimension k if and only if p is not an even integer.*

The hardest part of the theorem is item (b) which can be thought of as representing the two cases of intermediate dimension, namely $d = k = 3$ and $d \ge 3$, $k = 2$. These two cases require completely different methods ...

The geometry of the case $p = 2n$

Let Γ be a non-empty subset of $\mathbb{N}_0^d$ and suppose that λ is in $\Lambda(\Gamma)$, which is the coset generated by Γ. The *distance* from Γ to λ is

$$d(\Gamma, \lambda) := \inf \max \left(\sum_{m_{\gamma,\alpha}>0} m_{\gamma,\alpha}, -\sum_{m_{\gamma,\alpha}<0} m_{\gamma,\alpha} \right)$$

where the infimum is taken over all possible representations

$$\lambda = \gamma + \sum_{\substack{\alpha\in\Gamma \\ \alpha\neq\gamma}} m_{\gamma,\alpha}(\alpha - \gamma), \quad \gamma \in \Gamma.$$

For a non-negative integer n, the n-*extension* of Γ is

$$E_n(\Gamma) := \left\{ \lambda \in \Lambda(\Gamma) \cap \mathbb{N}_0^d \,:\, d(\Gamma, \lambda) \leq n \right\}.$$

An effective result

Theorem

Let $n \in \mathbb{N}$. *A set* Γ *in* $\mathbb{N}_0^d$ *is a contractive projection set for* $H^{2(n+1)}(\mathbb{T}^d)$ *if and only if* $E_n(\Gamma) = \Gamma$.

Example: $E_1(\Gamma) = \Gamma$ and $E_2(\Gamma) \neq \Gamma$

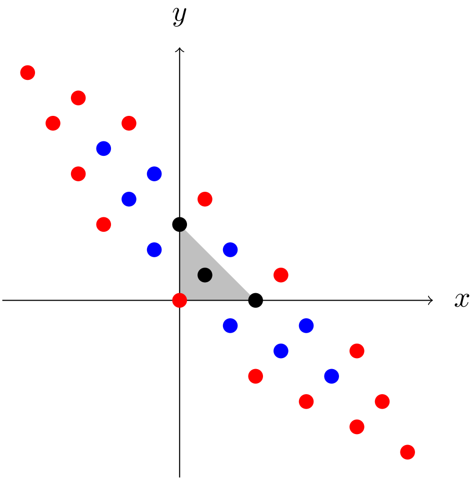

Figure: Points λ which satisfy $d(\Gamma, \lambda) = 1$ and $d(\Gamma, \lambda) = 2$ for $\Gamma = \{(3,0,0),(0,3,0),(1,1,1)\}$, in the plane $z = 3 - x - y$. The shaded triangle represents the intersection of this plane and the narrow cone $\mathbb{N}_0^d$.

Duality formulation for $1 \leq p < \infty$

Lemma

Fix $1 \leq p < \infty$ and $d \geq 1$. A set of frequencies Γ in $\mathbb{N}_0^d$ is a contractive projection set for $H^p(\mathbb{T}^d)$ if and only if

$$\int_{\mathbb{T}^d} |f(z)|^{p-2} f(z)\, \overline{z^\lambda}\, dm_d(z) = 0$$

for every f in $H^p(\mathbb{T}^d)$ of the form $f(z) = \sum_{\gamma \in \Gamma} a_\gamma z^\gamma$ and every λ in $\big(\Lambda(\Gamma) \cap \mathbb{N}_0^d\big) \setminus \Gamma$.

- The fact that Γ in $\mathbb{N}_0^d$ is a contractive projection set for $H^{2(n+1)}(\mathbb{T}^d)$ if and only if $E_n(\Gamma) = \Gamma$ is a geometric reformulation of this result.
- The case $1 \leq p < \infty$, $p \neq 2n$, follows almost immediately (next slide).

The case $1 \leq p < \infty$, p not even

If Γ is not the restriction of a coset in $\mathbb{Z}^d$ to $\mathbb{N}_0^d$, then there is some λ in $\big(\Lambda(\Gamma) \cap \mathbb{N}_0^d\big) \setminus \Gamma$. There is an affinely independent subset $\{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ of Γ which generates $\Lambda(\Gamma)$, where $n = \dim(\Lambda(\Gamma))$. Hence

$$\lambda = \gamma_0 + \sum_{j=1}^{n} m_j(\gamma_j - \gamma_0).$$

Set

$$f(z) := z^{\gamma_0} + \varepsilon \sum_{j=1}^{n} z^{\gamma_j}$$

for $0 < \varepsilon < 1/n$. We use the binomial series to express

$$\int_{\mathbb{T}^d} |f(z)|^{p-2} f(z)\, \overline{z^{\lambda}}\, dm_d(z)$$

as a non-trivial power series in ε, which is in conflict with the preceding lemma.

Key lemmas for the sufficiency part, $d \geq 3$ and $k = 2$

Lemma (Main lemma)

Fix $d \geq 3$ and let T be a set in $\mathbb{N}_0^d$ with $\dim(T) = 2$. Then the 2-completion of T is $\Lambda(T) \cap \mathbb{N}_0^d$.

To prove this result, we start from the special case of three points:

Lemma

Let T be a set of three affinely independent points in $\mathbb{N}_0^d$ for $d \geq 3$. Then the 2-completion of T is $\Lambda(T) \cap \mathbb{N}_0^d$.

The proofs are quite arithmetic. We prove the main lemma by a kind of Euclidean algorithm, starting from the three point lemma.

The case $d = k$

The extension problem is of a rather different nature when $d = k$. Indeed, our job is then mainly to reach ∞ inside the narrow cone. This is reflected in the following basic result.

Lemma

Let T be a subset of $\mathbb{N}_0^d$. If there are points α and β in $E_n^\infty(T)$ such that $\beta - \alpha$ is in $\mathbb{N}^d$, then

$$E_n^\infty(T) = E_1^\infty(T \cup \{\alpha, \beta\}) = \Lambda(T) \cap \mathbb{N}_0^d.$$

Here

$$E_n^\infty(T) := \bigcup_{k=1}^{\infty} E_n^k(T),$$

i.e., $E_n^\infty(T)$ is the smallest subset Γ of $\mathbb{N}_0^d$ such that $T \subset \Gamma$ and $E_n(\Gamma) = \Gamma$.

Key lemmas for the sufficiency part, $d = k = 2$ and $d = k = 3$

In view of the preceding lemma, all we need are the following two results.

Lemma

Let T be a set of three affinely independent points in $\mathbb{N}_0^2$. Then for every α in T there exists a point β in $E_1^\infty(T) \setminus \{\alpha\}$ such that $\beta - \alpha$ is in $\mathbb{N}^2$.

Lemma

Let T be a set of four affinely independent points in $\mathbb{N}_0^3$. Then for every α in T there exists a point β in $E_2^\infty(T) \setminus \{\alpha\}$ such that $\beta - \alpha$ is in $\mathbb{N}^3$.

The proof of both lemmas rely on a simple idea (next slide), but the proof of the latter is quite hard and requires a somewhat involved combinatorial argument.

Increasing iteratively the “negativity index”

Definition (Negativity index)

Given a set U of d linearly independent vectors $u = (u_1, \dots, u_d)$ in $\mathbb{Z}^d$, we define the *negativity index of* U as

$$\operatorname{ind}(U) := \sum_{j=1}^{d} \min\left(0, \min_{u \in U} u_j\right).$$

Proof idea: Successively change U by making 1- or 2-extensions of $\alpha + U$ to get to new vectors with a larger negativity index. For this to work, it is crucial that linear independence of the vectors of U be preserved during the course of the iteration!

Examples in “intermediate” dimensions

The necessity part of our main theorem are proved by finding suitable example sets.

- The example $\Gamma := \{(3,0,0),(0,3,0),(1,1,1)\}$ settles the necessity of the condition for $d \geq 3$ and $k = 2$.
- The example $\Gamma := \{(4,0,0),(0,4,0),(0,0,4),(1,1,1)\}$ settles the necessity of the condition for $d = k = 3$.

Example in “high” dimensions

- The example ($n \geq 3$)

  $$\{(n, 1, 0, 1), (n + 1, 0, 1, 0), (0, 0, n + 1, 0), (0, 0, 0, n + 1), (0, n + 1, 0, 0)$$

  settles the necessity of the condition for $d \geq 4$ and $4 \leq k \leq d$ in part (c). (The set equals its n-extension.)

The $n + 1$-extension of the set is the full coset intersected with the narrow cone.

Examples of exotic linear operators on $H^p(\mathbb{T}^\infty)$

Using the above example sets, it is easy to cook up an explicit linear operator to arrive at the following result.

Theorem

Fix an integer $n \geq 1$. There is a linear operator T_n that is densely defined on $H^p(\mathbb{T}^\infty)$ for every $1 \leq p \leq \infty$, and that extends to a bounded operator on $H^p(\mathbb{T}^\infty)$ if and only if $p = 2, 4, \ldots, 2(n+1)$.

This result exemplifies quite strikingly the impossibility of interpolating between Hardy spaces on the infinite-dimensional torus, as studied in depth in a recent paper of Bayart and Mastylo (2019).

Is there an even more exotic linear operator on $H^p(\mathbb{T}^\infty)$?

Question

Is there a linear operator T_∞ that is densely defined on $H^p(\mathbb{T}^\infty)$ for every $1 \le p \le \infty$, and that extends to a bounded operator on $H^p(\mathbb{T}^\infty)$ if and only if $p = 2n$, $n = 1, 2, \ldots$?

This question is related to an old problem in the theory of Hardy spaces of Dirichlet series: For which p is there an absolute constant C_p such that

$$\int_0^1 |F(1/2 + it)|^p dt \le C_p \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(it)|^p dt \qquad (1)$$

for all Dirichlet polynomials $F(s) = a_1 + a_2 2^{-s} \cdots a_n N^{-s}$? This is true when $p = 2n$ (easy) and is known to fail when $0 < p < 2$ by a recent theorem of Harper (2020) (a deep result).

Failure of (1) for $p \neq 2n$, $p > 2$, would, via the Bohr lift, yield such an exotic T_∞.