Introduction to multiplicative Toeplitz and Hankel operators

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Overview

Toeplitz $\{t(j-k)\}_{j,k=0}^{\infty}$ in $\ell^2(\mathbb{Z}_+)$	Hankel $\{h(j+k)\}_{j,k=0}^{\infty}$ in $\ell^2(\mathbb{Z}_+)$
Multiplicative Toeplitz $\{t(n/m)\}_{n,m=1}^{\infty}$ in $\ell^2(\mathbb{N})$	Multiplicative Hankel (=Helson) $\{h(nm)\}_{n,m=1}^{\infty}$ in $\ell^2(\mathbb{N})$

- Realisation in Hardy space
- Boundedness
- Some aspects of spectral theory
- Connection to Dirichlet series

Additive Toeplitz and Hankel operators: definitions, examples

Symbol:

$$\varphi \in L^\infty(\mathbb{T}),\ \varphi(z) = \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k) z^k$$

Toeplitz matrix: $\{\widehat{\varphi}(j-k)\}_{j,k=0}^{\infty}$
Hankel matrix: $\{\widehat{\varphi}(j+k)\}_{j,k=0}^{\infty}$

Realisation in $H^2(\mathbb{T})$: $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$,

$$P : L^2 \to H^2,\ \sum_{k=-\infty}^{\infty} \widehat{f}(k) z^k \mapsto \sum_{k=0}^{\infty} \widehat{f}(k) z^k$$

$T(\varphi)f = P(\varphi \cdot f)$,
$H(\varphi)f = P(\varphi \cdot f(\overline{z}))$

NB: $H(\varphi)$ depends only on $P\varphi$

Example

$\varphi \in H^\infty$: $T(\varphi)f = \varphi \cdot f$

Example

$\varphi(z) = \overline{z}$: $T(\overline{z})f = P(\overline{z}f) = S^* f$

Example

$\widehat{\varphi}(k) = \frac{1}{k+1}$: $\{H(\varphi)\} = \{\frac{1}{j+k+1}\}_{j,k=0}^{\infty}$

Questions

- What is the class of all bounded/compact/trace class/finite rank $T(\varphi)/H(\varphi)$?
- What is the spectrum of $T(\varphi)/H(\varphi)$ for a given φ?
- What is the behaviour of the $N \times N$ truncations of $\{T(\varphi)\}/\{H(\varphi)\}$ as $N \to \infty$? (norms, eigenvalues, determinants etc)

Boundedness

Toeplitz

Theorem

$T(\varphi)$ is bdd iff $\varphi \in L^\infty$; $\|T(\varphi)\| = \|\varphi\|_{L^\infty}$

Proof.

$\|T(\varphi)f\| = \|P(\varphi f)\| \leq \|\varphi\|_{L^\infty}\|f\|$

$M(\varphi)f = \varphi f$ in $L^2(\mathbb{T})$, $\|M(\varphi)\| = \|\varphi\|_{L^\infty}$

$\|M(\varphi)\| \leq \liminf_{N\to\infty}\|M_N(\varphi)\| = \|T(\varphi)\|$

□

Hankel

- $\|H(\varphi)\| \leq \|\varphi\|_{L^\infty}$
- $H(\varphi) = H(\psi)$ if $P\psi = P\varphi$, and so $\|H(\varphi)\| \leq \inf\{\|\psi\|_{L^\infty} : P\psi = P\varphi\}$
- Z.Nehari 1958: "="
  $H(\varphi)$ is bdd iff $\exists\psi \in L^\infty$: $P\psi = P\varphi$

Hilbert's matrix

Hankel: compactness, finite rank

Toeplitz: $T(\varphi)$ is compact iff $\varphi = 0$

Hankel: compactness

- Nehari 1958:
  $H(\varphi)$ is bounded iff $\exists \psi \in L^\infty(\mathbb{T})$: $P\psi = P\varphi$
- P.Hartman 1959:
  $H(\varphi)$ is compact iff $\exists \psi \in C(\mathbb{T})$: $P\psi = P\varphi$

Hankel: Finite rank property

L.Kronecker 1881:
$H(\varphi)$ has finite rank iff φ is rational

Example

Hankel: examples

- Hilbert's matrix again: $\{H\} = \{(j+k+1)^{-1}\}_{j,k=0}^{\infty}$
- M.Rosenblum 1958: $\sigma(H) = [0,\pi]$, purely a.c., multiplicity=1
- Explicit diagonalisation (continuous dual Hahn polynomials)
- $h(j) = O(1/j) \Rightarrow \{h(j+k)\}$ is bounded
- $h(j) = o(1/j) \Rightarrow \{h(j+k)\}$ is compact
- H.Widom 1966: let $h(j) = (j+k+1)^{-1-\delta}$, $\delta > 0$

  then the singular values of $\{h(j+k)\}$ obey $s_n = e^{-\pi\sqrt{2\delta n}+o(\sqrt{n})}$, $n \to \infty$
- Pushnitski+Yafaev 2015: let $h(j) = j^{-1}(\log j)^{-\alpha}$, $\alpha > 0$, $j \geq 2$

  then the singular values of $\{h(j+k)\}$ obey $s_n = c_\alpha n^{-\alpha} + o(n^{-\alpha})$

Toeplitz: essential spectrum, a.c. spectrum, finite truncations

Essential spectrum, a.c. spectrum

- $\varphi \in C(\mathbb{T})$: $\sigma_{\text{ess}}(T(\varphi)) = \varphi(\mathbb{T})$
- Well developed Fredholm theory
- $\varphi = \overline{\varphi}$:
  $\sigma(T(\varphi)) = [\operatorname{ess\,inf} \varphi, \operatorname{ess\,sup} \varphi]$,
  purely a.c.
  (Hartman-Wintner, Putnam, M.Rosenblum 1950-1965)

Finite truncations

$\{T(\varphi)\}_N$: $N \times N$ truncations

- Connection with orthogonal polynomials
- analysis of $\det\{T(\varphi)\}_N$ as $N \to \infty$
- Szegő 1915: for $\Delta \subset \mathbb{R}$,

$$\lim_{N\to\infty} \frac{\#\{n : \lambda_n(\{T(\varphi)\}_N) \in \Delta\}}{N} = m\{z : \varphi(z) \in \Delta\}$$

Summary

- Toeplitz and Hankel operators: in $\ell^2(\mathbb{Z}_+)$ or in $H^2(\mathbb{T})$
- Key point in study: map $\varphi \mapsto$ spectral properties of $T(\varphi)/H(\varphi)$
- Toeplitz: never compact; φ is related to essential spectrum and spectral density of $T(\varphi)$
- Hankel: sometimes compact; "degree of compactness" of $H(\varphi)$ is related to decay rate of $\widehat{\varphi}$

Multi-variable Toeplitz and Hankel operators: definitions

- Notation: $z = (z_1, z_2, \dots, z_d) \in \mathbb{T}^d$, $d \geq 1$; $z^k := z_1^{k_1} \cdots z_d^{k_d}$, $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$
- $\{z^k\}_{k \in \mathbb{Z}_+^d}$ is ONB in $H^2(\mathbb{T}^d)$
- For $f \in H^2(\mathbb{T}^d)$, write $\widehat{f}(k) = \langle f, z^k \rangle$, $k \in \mathbb{Z}^d$
- $P_d : L^2(\mathbb{T}^d) \to H^2(\mathbb{T}^d)$
- For $\varphi \in L^\infty(\mathbb{T}^d)$ define

$$T_d(\varphi) f = P_d(\varphi \cdot f),$$
$$H_d(\varphi) f = P_d(\varphi \cdot f(\overline{z}))$$

- Matrices:

$$\langle T_d(\varphi) z^j, z^k \rangle = \widehat{\varphi}(j - k),$$
$$\langle H_d(\varphi) z^j, z^k \rangle = \widehat{\varphi}(j + k)$$

Multi-variable Toeplitz and Hankel operators: boundedness, spectral properties

Boundedness

- $\|T_d(\varphi)\| = \|\varphi\|_{L^\infty}$
- $\|H_d(\varphi)\| \leq \inf\{\|\psi\|_{L^\infty} : P_d\psi = P_d\varphi\}$
- Analogue of Nehari's theorem: $\inf\{\|\psi\|_{L^\infty} : P_d\psi = P_d\varphi\} \leq C_d\|H_d(\varphi)\|$
  Lacey-Ferguson 2002, $d = 2$; Lacey-Terwilliger 2009, $d > 2$
- Ortega-Cerda-Seip 2012: $C_d \to \infty$ as $d \to \infty$

Other

- Fredholm theory for $T_d(\varphi)$ (at least for continuous φ)
- Analogue of Szegő's theorem for $T_d(\varphi)$. $\{T_d(\varphi)\}_\omega = \{\widehat{\varphi}(j-k)\}_{j,k\in\omega}$

$$\lim_{\#\omega\to\infty} \frac{\#\{n : \lambda_n(\{T_d(\varphi)\}_\omega) \in \Delta\}}{\#\omega} = m_d\{z \in \mathbb{T}^d : \varphi(z) \in \Delta\}$$

- M.Rosenblum 1973: $T_d(\varphi) = T_d(\varphi)^*$ are not necessarily a.c.
- Finite rank $H_d(\varphi)$ described by S.Power 1977 (rational symbols)
- A class of explicit compact $H_d(\varphi)$ with spectral asymptotics: C.Tantalakis (in progress)

Multiplicative Toeplitz and Hankel operators and Dirichlet series

Function spaces

$$\mathscr{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} f_n n^{-s}, \sum_{n=1}^{\infty} |f_n|^2 < \infty \right\}$$

Besicovitch space B^2 of almost-periodic functions: closure of polynomials $f(t) = \sum_n a_n e^{i\lambda_n t}$ in the norm $\|f\|_{B^2}^2 = \sum_n |a_n|^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt$

$\mathscr{H}^2 \subset B^2$; $P_\infty : B^2 \to \mathscr{H}^2$

$\{t(n/m)\}_{n,m=1}^{\infty}$, $\{h(nm)\}_{n,m=1}^{\infty}$ in $\ell^2(\mathbb{N})$

Symbol: $\varphi(t) = \sum_{q\in\mathbb{Q}_+} \widehat{\varphi}(q) q^{-it}$

$T_\infty(\varphi)f = P_\infty(\varphi \cdot f)$, $H_\infty(\varphi)f = P_\infty(\varphi \cdot f(-it))$

$$\langle T_\infty(\varphi) n^{-it}, m^{-it} \rangle_{\mathscr{H}^2} = \widehat{\varphi}(m/n)$$
$$\langle H_\infty(\varphi) n^{-it}, m^{-it} \rangle_{\mathscr{H}^2} = \widehat{\varphi}(mn)$$

Example: Multipliers on $\mathscr{H}^2$

$\varphi \in \mathscr{H}^\infty$: $T_\infty(\varphi) = \|\varphi\|_{\mathscr{H}^\infty}$

Toeplitz 1938, Hedenmalm-Lindqvist-Seip 1997

Bemerkung. Bedient man sich der Sprechweise der Theorie der unendlich vielen Variablen, so ist $D_n(x, \bar{x})$ nichts anderes als der n-te Abschnitt der Bilinearform von unendlich vielen Variablen

$$\mathrm{D}(x, y) = c_1(x_1 y_1 + x_2 y_2 + \cdots) + c_2(x_1 y_2 + x_2 y_4 + \cdots) \\ + c_3(x_1 y_3 + x_2 y_6 + \cdots) + \cdots,$$

nur dass in den n-ten Abschnitt $D_n(x, y)$ hiervon noch $y_1 = \bar{x}_1, \cdots, y_n = \bar{x}_n$ gesetzt ist. Nun besagt ein einfacher Hilfssatz [3] über Bilinearformen von endlich vielen Veränderlichen $A(x, y)$, dass das Maximum von $|A(x, y)|$ für $\Sigma x_a \bar{x}_a = 1$, $\Sigma y_a \bar{y}_a = 1$ höchstens gleich dem Doppelten des Maximums von $A(x, \bar{x})$ für $\Sigma x_a \bar{x}_a = 1$ ist; also liegt im vorliegenden Falle $|D_n(x, y)|$ unter $2M$, und Theorem I kann auch so formuliert werden:

Befriedigt eine Belegung die eingangs dieses Paragraphen aufgeführten Voraussetzungen, so soll die aus ihren Hadamardschen Koeffizienten c_n aufgebaute Bilinearform $D(x, y)$ als die zu der Belegung gehörige „*D-Form*" bezeichnet werden. *Ist die Belegung beschränkt, $|f(ti)| \leqq M$, so ist die D-Form im Hilbertschen Sinne beschränkt, und ihre obere Schranke liegt unter $2M$.*

Die Koeffizientenmatrix einer solchen D-Form hat die Eigentümlichkeit, dass die Stellen, die den nämlichen Koeffizienten tragen, je eine gerade Linie erfüllen und dass alle diese geraden Linien ein Strahlenbüschel bilden mit endlichem Zentrum: [4]

$$D = \left\|\begin{matrix}
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & \cdots \\
0 & c_1 & 0 & c_2 & 0 & c_3 & 0 & c_4 & 0 & c_5 & 0 & c_6 & \cdots \\
0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & \cdots \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & \cdots \\
0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & c_2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots
\end{matrix}\right\|.$$

[3] Vgl. eine inzwischen in der *Mathematischen Zeitschrift* (Bd. 2 (1918), S. 187)

Multiplicative Toeplitz and Hankel operators in $H^2(\mathbb{T}^\infty)$

Notation

- $\mathbb{N} \ni n = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}\cdots =: p^\alpha$,
  $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{Z}_+^{(\infty)}$
- $\mathbb{N} \simeq \mathbb{Z}_+^{(\infty)}$, $\ell^2(\mathbb{N}) \simeq \ell^2(\mathbb{Z}_+^{(\infty)})$

Set $n = p^\alpha$, $m = p^\beta$:

$$\{t(n/m)\}_{n,m\in\mathbb{N}} \simeq \{t(p^{\alpha-\beta})\}_{\alpha,\beta\in\mathbb{Z}_+^{(\infty)}}$$
$$\{h(nm)\}_{n,m\in\mathbb{N}} \simeq \{h(p^{\alpha+\beta})\}_{\alpha,\beta\in\mathbb{Z}_+^{(\infty)}}$$

Infinite multi-torus

$\mathbb{T}^\infty = \mathbb{T} \times \mathbb{T} \times \dots$ compact Abelian group, $\exists$ invariant measure m_∞

$L^2(\mathbb{T}^\infty)$, ONB: $\{z^\alpha\}_{\alpha\in\mathbb{Z}^{(\infty)}}$
$H^2(\mathbb{T}^\infty) \subset L^2(\mathbb{T}^\infty)$, ONB: $\{z^\alpha\}_{\alpha\in\mathbb{Z}_+^{(\infty)}}$

$P_\infty : L^2(\mathbb{T}^\infty) \to H^2(\mathbb{T}^\infty)$

Toeplitz and Hankel operators

For a symbol $\varphi \in L^\infty(\mathbb{T}^\infty)$, define operators in $H^2(\mathbb{T}^\infty)$:

$$T_\infty(\varphi)f = P_\infty(\varphi \cdot f),$$
$$H_\infty(\varphi)f = P_\infty(\varphi \cdot f(\overline{z}))$$

$$\begin{array}{ccc} \ell^2(\mathbb{N}) & \longrightarrow & \ell^2(\mathbb{Z}_+^{(\infty)}) \\ \downarrow & & \downarrow \\ \mathscr{H}^2 & \longrightarrow & H^2(\mathbb{T}^\infty) \end{array}$$

Boundedness and spectral properties

$T_\infty(\varphi)f = P_\infty(\varphi \cdot f)$
$H_\infty(\varphi)f = P_\infty(\varphi \cdot f(\overline{z}))$

Boundedness

- $\|T_\infty(\varphi)\| = \|\varphi\|_{L^\infty(\mathbb{T}^\infty)}$
- $\|H_\infty(\varphi)\| \leq \|\varphi\|_{L^\infty(\mathbb{T}^\infty)}$, but Nehari's theorem is false! Ortega-Cerda+Seip constructed a sequence $\{\varphi_n\}_{n=1}^\infty$: $\|H_\infty(\varphi_n)\| \leq C$, $\inf\{\|\psi\|_{L^\infty} : P_\infty\psi = P_\infty\varphi_n\} \to \infty$ as $n \to \infty$.
- Helson 2006: $H_\infty(\varphi) \in \mathfrak{S}_2 \Rightarrow \exists$ bounded symbol
- Brevig-Perfekt 2015: false for $\mathfrak{S}_p$, $p > 5.74$

Other

- Compactness? Schatten classes?
- $H_\infty(\varphi)$ of finite rank: Perfekt-Pushnitski 2018

Work on multiplicative Toeplitz/Hankel operators to date

- General questions: boundedness, Nehari's theorem, etc.
- Arithmetic multiplicative Toeplitz matrices, e.g. $t(n/m) = \dfrac{(\gcd(n,m))^{2\sigma}}{(nm)^{\sigma}}$

  Often are multiplicative: $t(p^{\alpha}) = \prod_{p_i} t(p_i^{\alpha_i})$

  Multiplicativity leads to tensor product representation $\otimes_p \{t(p^{j-k})\}_{j,k\in\mathbb{Z}_+}$ (Hilberdink)

- Multiplicative Hankel matrices: restrictions $h(nm)$ where $h(xy)$ is a "nice" kernel

  Comparision with integral operators $f \mapsto \int_1^{\infty} h(xy) f(y) dy$

  See Naz Miheisi's talk

The multiplicative Hilbert matrix

- $\{H_\infty(\varphi)\} = \{(nm)^{-\frac{1}{2}}(\log nm)^{-1}\}_{n,m=2}^{\infty}$
- Brevig-Perfekt-Seip-Siskakis-Vukotic 2016, Perfekt-Pushnitski 2018
- Heuristics: related to $f \mapsto \int_2^\infty (st)^{-\frac{1}{2}}(\log st)^{-1} f(s)ds$ in $L^2(2,\infty)$
- Change of variable: $F \mapsto \int_{\log 2}^\infty (x+y)^{-1}F(y)dy$ in $L^2(\log 2,\infty)$
- "1/2 of the Carleman operator"
- $H_\infty(\varphi)$ has purely a.c. spectrum $[0,\pi]$, multiplicity=1.
- No explicit diagonalisation known!
- $H_\infty(\varphi)$ is unitarily equivalent to the integral operator in $L^2(0,\infty)$ with the Hankel kernel $\zeta(x+y+1)-1$
- Dirichlet symbol: $\varphi(t) = \sum_{n=2}^\infty (\log n)^{-1} n^{-\frac{1}{2}-it}$, $\varphi'(t) = -i\zeta(\frac{1}{2}+it)$
- Does it have a bounded symbol?

Truncations of multiplicative Toeplitz operators I: Følner sequences

- $\varphi \in L^\infty(\mathbb{T}^\infty)$, consider $T_\infty(\varphi)$
- $\omega \subset \mathbb{N}$: $\{T_\infty(\varphi)\}_\omega = \{\widehat{\varphi}(n/m)\}_{n,m\in\omega}$
- Szegő type theorem:

$$\lim_{N\to\infty} \frac{\#\{n : \lambda_n(\{T_\infty(\varphi)\}_{\omega_N}) \in \Delta\}}{\#\omega_N} = m_\infty\{z \in \mathbb{T}^\infty : \varphi(z) \in \Delta\}$$

  if $\{\omega_N\}$ is a multiplicative Følner sequence:

$$\lim_{\#\omega_N\to\infty} \frac{\#\{k \in \omega_N : nk \in \omega_N\}}{\#\omega_N} = 1, \quad \forall n \in \mathbb{N}$$

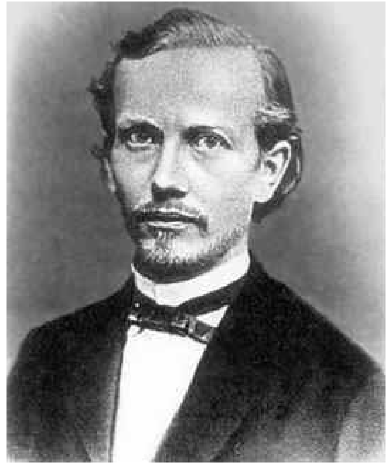
- Bédos 1997, Nikolski+Pushnitski 2020
- $\omega_N = \{1, \dots, N\}$ is NOT multiplicative Følner!
- A multiplicative Følner sequence:
  $\omega_N = \{p^\alpha : 0 \le \alpha_j \le A_j^{(N)}\}$, $\forall j$: $A_j^{(N)} \to \infty$ as $N \to \infty$

Truncations of multiplicative Toeplitz operators II: truncations to $\{1,\ldots,N\}$

- $\{T_\infty(\varphi)\}_N = \{\widehat{\varphi}(n/m)\}_{n,m=1}^N$
- Fix $\sigma > 0$, let $\varphi_\sigma(t) = |\zeta(\sigma+it)|^2$; consider $T_\infty(\varphi_\sigma)$.
- $[T_\infty(\varphi_\sigma)]_{n,m} = \zeta(2\sigma)\dfrac{(\gcd(n,m))^{2\sigma}}{(nm)^\sigma}$, $\sigma > 1$
- For $\sigma \le 1$, $|\zeta(\sigma+it)|^2$ is unbounded, and so is $T_\infty(\varphi_\sigma)$.
- What about the truncation $\{T_\infty(\varphi_\sigma)\}_N$?
  Is it related to $|\zeta(\sigma+it)|^2$ for $|t| \le T = T(N)$?
- Norms and singular values of $\{T_\infty(\varphi_\sigma)\}_N$ as $N \to \infty$:
  Lindqvist-Seip 1998, Hilberdink 2009, Bondarenko-Hilberdink-Seip 2016, Aistleitner 2016
- Sample result for $\frac{1}{2} \le \sigma \le 1$:

$$\sup_{|t|\le T} |\zeta(\sigma+it)|^2 \ge \|\{T_\infty(\varphi_\sigma)\}_N\| + \text{small error}, \quad N = T^{\frac{2}{3}(\sigma-\frac{1}{2})-\varepsilon}$$

Otto Toeplitz (1881-1940)



Hermann Hankel (1839-1873)