

Introduction to multiplicative Toeplitz and Hankel operators

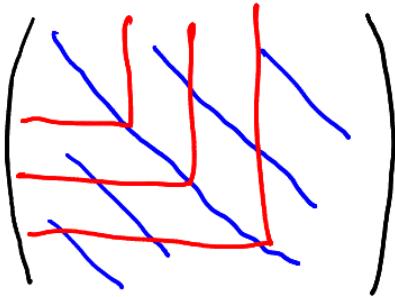
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Overview

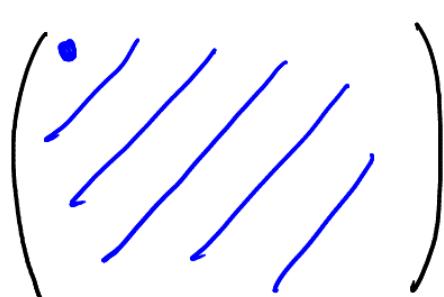
Toeplitz

$\{t(j - k)\}_{j,k=0}^{\infty}$ in $\ell^2(\mathbb{Z}_+)$



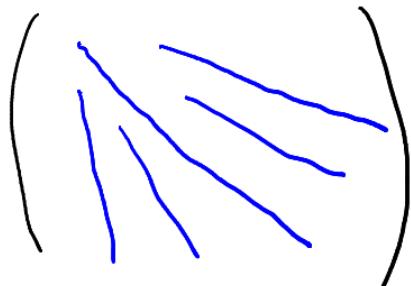
Hankel

$\{h(j + k)\}_{j,k=0}^{\infty}$ in $\ell^2(\mathbb{Z}_+)$



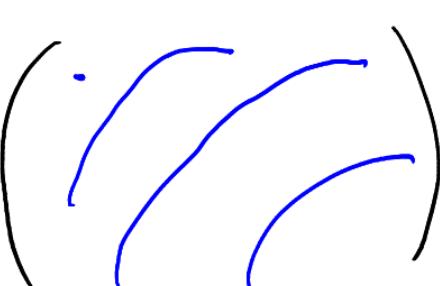
Multiplicative Toeplitz

$\{t(n/m)\}_{n,m=1}^{\infty}$ in $\ell^2(\mathbb{N})$



Multiplicative Hankel (=Helson)

$\{h(nm)\}_{n,m=1}^{\infty}$ in $\ell^2(\mathbb{N})$



- Realisation in Hardy space
- Boundedness
- Some aspects of spectral theory
- Connection to Dirichlet series

Additive Toeplitz and Hankel operators: definitions, examples

Symbol:

$$\varphi \in L^\infty(\mathbb{T}), \varphi(z) = \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k) z^k, |z|=1$$

Toeplitz matrix: $\{\widehat{\varphi}(j-k)\}_{j,k=0}^{\infty}$

Hankel matrix: $\{\widehat{\varphi}(j+k)\}_{j,k=0}^{\infty}$

Realisation in $H^2(\mathbb{T})$: $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$,

$$P : L^2 \rightarrow H^2, \sum_{k=-\infty}^{\infty} \widehat{f}(k) z^k \mapsto \sum_{k=0}^{\infty} \widehat{f}(k) z^k$$

$$\left. \begin{array}{l} T(\varphi)f = P(\varphi \cdot f), \\ H(\varphi)f = P(\varphi \cdot f(\bar{z})) \end{array} \right\} f \in \mathcal{H}^2(\mathbb{T})$$

$$\begin{aligned} \langle \varphi(\varphi \bar{z}^j), z^k \rangle &= \langle \varphi \bar{z}^j, z^k \rangle \\ &= \langle \varphi, z^{k+j} \rangle = \varphi(k+j) \end{aligned}$$

NB: $H(\varphi)$ depends only on $P\varphi$

Example

$$\varphi \in H^\infty: T(\varphi)f = \varphi \cdot f$$

Example

$$\varphi(z) = \bar{z}: T(\bar{z})f = P(\bar{z}f) = S^*f$$

Example

$$\widehat{\varphi}(k) = \frac{1}{k+1}: \{H(\varphi)\} = \{\frac{1}{j+k+1}\}_{j,k=0}^{\infty}$$

Questions

$$\varphi \mapsto T(\varphi)$$

$$\varphi \mapsto H(\varphi)$$

- What is the class of all bounded/compact/trace class/finite rank $T(\varphi)/H(\varphi)$?
- What is the spectrum of $T(\varphi)/H(\varphi)$ for a given φ ?
- What is the behaviour of the $N \times N$ truncations of $\{T(\varphi)\}/\{H(\varphi)\}$ as $N \rightarrow \infty$?
(norms, eigenvalues, determinants etc)

Boundedness

Toeplitz

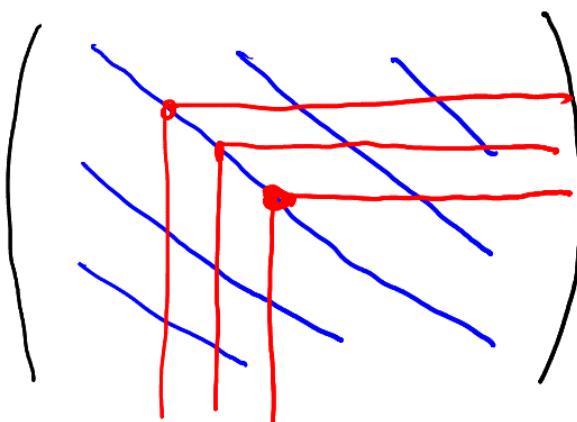
Theorem

$T(\varphi)$ is bdd iff $\varphi \in L^\infty$; $\|T(\varphi)\| = \|\varphi\|_{L^\infty}$

$$\|T(\varphi)f\| = \|P(\varphi f)\| \leq \|\varphi\|_{L^\infty} \|f\|$$

$$M(\varphi)f = \varphi \cdot f \text{ in } L^2(\pi)$$

$$\|M(\varphi)\| = \|\varphi\|_{L^\infty}$$



Hankel

- $\|H(\varphi)\| \leq \|\varphi\|_{L^\infty}$
- $H(\varphi) = H(\psi)$ if $P\psi = P\varphi$, and so $\|H(\varphi)\| \leq \inf\{\|\psi\|_{L^\infty} : P\psi = P\varphi\}$
- Z.Nehari 1958: “=”
 $H(\varphi)$ is bdd iff $\exists \psi \in L^\infty: P\psi = P\varphi$

Hilbert's matrix

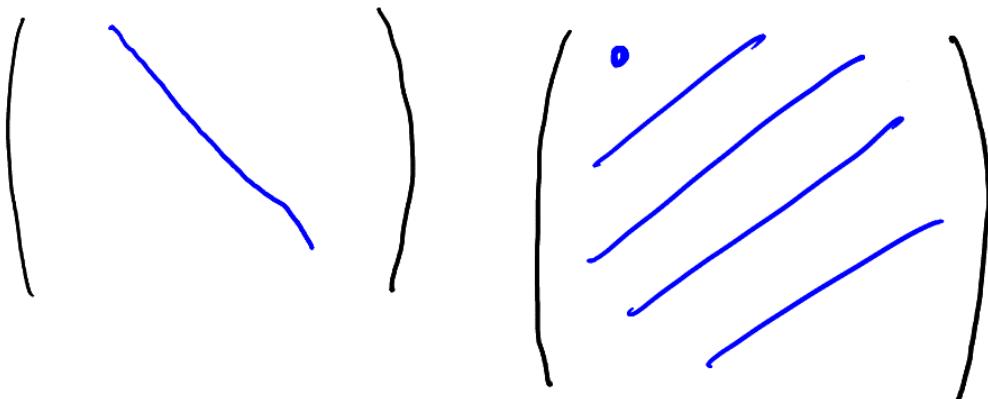
$$\widehat{\phi}(k) = \frac{1}{k+1}, k \geq 0$$
$$\sum_{k=0}^{\infty} \frac{1}{k+1} z^k - \sum_{k=1}^{\infty} \frac{1}{k+1} z^k \in L^\infty$$

unbdd

ψ

Hankel: compactness, finite rank

Toeplitz: $T(\varphi)$ is compact iff $\varphi = 0$



Hankel: compactness

- Nehari 1958:
 $H(\varphi)$ is bounded iff $\exists \psi \in L^\infty(\mathbb{T}): P\psi = P\varphi$
- P.Hartman 1959:
 $H(\varphi)$ is compact iff $\exists \psi \in C(\mathbb{T}): P\psi = P\varphi$

Hankel: Finite rank property

L.Kronecker 1881:

$H(\varphi)$ has finite rank iff φ is rational

Example

$$\varphi(z) = \frac{1}{1-\alpha z}, \quad 0 < \alpha < 1$$

$$= \sum_{k=0}^{\infty} \alpha^k z^k,$$

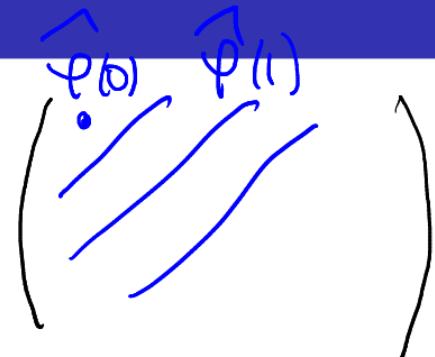
$$\{\alpha^{k+j}\} = \langle \cdot, A \rangle A$$

$$A = (\cancel{1}, \alpha, \alpha^2, \dots)$$

$$\left(\frac{1}{1-\alpha z} \right)^k$$

Hankel: examples

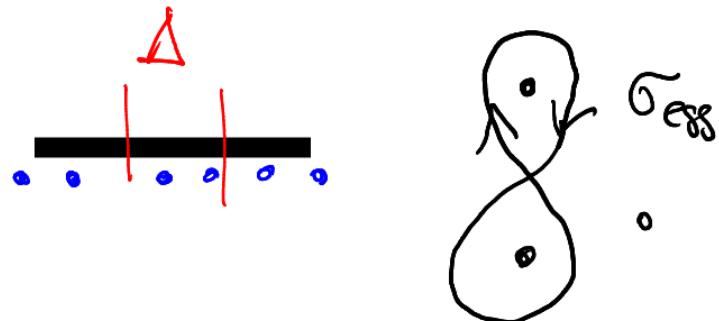
- Hilbert's matrix again: $\{H\} = \{(j+k+1)^{-1}\}_{j,k=0}^{\infty}$
- M.Rosenblum 1958: $\sigma(H) = [0, \pi]$, purely a.c., multiplicity=1
- Explicit diagonalisation (continuous dual Hahn polynomials)
- $h(j) = O(1/j) \Rightarrow \{h(j+k)\}$ is bounded
- $h(j) = o(1/j) \Rightarrow \{h(j+k)\}$ is compact
- H.Widom 1966: let $h(j) = (j + \cancel{M} + 1)^{-1-\delta}$, $\delta > 0$
then the singular values of $\{h(j+k)\}$ obey $s_n = e^{-\pi\sqrt{2\delta n} + o(\sqrt{n})}$, $n \rightarrow \infty$
- Pushnitski+Yafaev 2015: let $h(j) = j^{-1}(\log j)^{-\alpha}$, $\alpha > 0$, $j \geq 2$
then the singular values of $\{h(j+k)\}$ obey $s_n = c_{\alpha}n^{-\alpha} + o(n^{-\alpha})$



Toeplitz: essential spectrum, a.c. spectrum, finite truncations

Essential spectrum, a.c. spectrum

- $\varphi \in C(\mathbb{T})$: $\sigma_{\text{ess}}(T(\varphi)) = \varphi(\mathbb{T})$
- Well developed Fredholm theory
- $\varphi = \overline{\varphi}$:
 $\sigma(T(\varphi)) = [\text{ess inf } \varphi, \text{ess sup } \varphi]$,
purely a.c.
(Hartman-Wintner, Putnam,
M.Rosenblum 1950-1965)



Finite truncations

$\{T(\varphi)\}_N$: $N \times N$ truncations

- Connection with orthogonal polynomials
- analysis of $\det\{T(\varphi)\}_N$ as $N \rightarrow \infty$
- Szegő 1915: for $\Delta \subset \mathbb{R}$,

$$\lim_{N \rightarrow \infty} \frac{\#\{n : \lambda_n(\{T(\varphi)\}_N) \in \Delta\}}{N} = m\{z : \varphi(z) \in \Delta\}$$

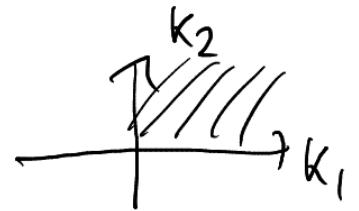
Summary

- Toeplitz and Hankel operators: in $\ell^2(\mathbb{Z}_+)$ or in $H^2(\mathbb{T})$
- Key point in study: map $\varphi \mapsto$ spectral properties of $T(\varphi)/H(\varphi)$
- Toeplitz: never compact; φ is related to essential spectrum and spectral density of $T(\varphi)$
- Hankel: sometimes compact; “degree of compactness” of $H(\varphi)$ is related to decay rate of $\widehat{\varphi}$

Multi-variable Toeplitz and Hankel operators: definitions

- Notation: $z = (z_1, z_2, \dots, z_d) \in \mathbb{T}^d$, $d \geq 1$; $z^k := z_1^{k_1} \cdots z_d^{k_d}$, $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$
- $\{z^k\}_{k \in \mathbb{Z}_+^d}$ is ONB in $H^2(\mathbb{T}^d)$
- For $f \in H^2(\mathbb{T}^d)$, write $\widehat{f}(k) = \langle f, z^k \rangle$, $k \in \mathbb{Z}^d$
- $P_d : L^2(\mathbb{T}^d) \rightarrow H^2(\mathbb{T}^d)$
- For $\varphi \in L^\infty(\mathbb{T}^d)$ define

$$\sum_{k \in \mathbb{Z}_+^d} \widehat{f}(k) z^k \mapsto \sum_{k \in \mathbb{Z}_+^d} \widehat{f}(k) z^k$$



$$\begin{aligned} T_d(\varphi)f &= P_d(\varphi \cdot f), \\ H_d(\varphi)f &= P_d(\varphi \cdot f(\bar{z})) \end{aligned} \quad \left. \begin{array}{c} f \in H^2(\mathbb{T}^d) \end{array} \right\}$$

- Matrices:

$$\begin{aligned} \langle T_d(\varphi)z^j, z^k \rangle &= \widehat{\varphi}(j - k), \\ \langle H_d(\varphi)z^j, z^k \rangle &= \widehat{\varphi}(j + k) \end{aligned} \quad j, k \in \mathbb{Z}_+^d$$

Multi-variable Toeplitz and Hankel operators: boundedness, spectral properties

Boundedness

- $\|T_d(\varphi)\| = \|\varphi\|_{L^\infty}$
- $\|H_d(\varphi)\| \leq \inf\{\|\psi\|_{L^\infty} : P_d\psi = P_d\varphi\}$
- Analogue of Nehari's theorem: $\inf\{\|\psi\|_{L^\infty} : P_d\psi = P_d\varphi\} \leq C_d \|H_d(\varphi)\|$ } Question?
Lacey-Ferguson 2002, $d = 2$; Lacey-Terwilliger 2009, $d > 2$
- Ortega-Cerda-Seip 2012: $C_d \rightarrow \infty$ as $d \rightarrow \infty$

Other

- Fredholm theory for $T_d(\varphi)$ (at least for continuous φ)
- Analogue of Szegő's theorem for $T_d(\varphi)$. $\{T_d(\varphi)\}_\omega = \{\widehat{\varphi}(j - k)\}_{j,k \in \omega}$ $\varphi = \overline{\varphi}$ $\omega \subset \mathbb{Z}_+^d$

$$\lim_{\#\omega \rightarrow \infty} \frac{\#\{n : \lambda_n(\{T_d(\varphi)\}_\omega) \in \Delta\}}{\#\omega} = m_d\{z \in \mathbb{T}^d : \varphi(z) \in \Delta\}$$

- M.Rosenblum 1973: $T_d(\varphi) = T_d(\varphi)^*$ are not necessarily a.c.
- Finite rank $H_d(\varphi)$ described by S.Power 1977 (rational symbols)
- A class of explicit compact $H_d(\varphi)$ with spectral asymptotics: C.Tantalakis (in progress)

Multiplicative Toeplitz and Hankel operators and Dirichlet series

Function spaces

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} f_n n^{-s}, \sum_{n=1}^{\infty} |f_n|^2 < \infty \right\}$$

Besicovitch space B^2 of almost-periodic functions: closure of polynomials

$$f(t) = \sum_n a_n e^{i\lambda_n t} \text{ in the norm}$$

$$\|f\|_{B^2}^2 = \sum_n |a_n|^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt$$

$$\mathcal{H}^2 \subset B^2; P_{\infty} : B^2 \rightarrow \mathcal{H}^2$$

$$\sum f_n n^{-it} = \sum f_n e^{-it \underline{\log n}}$$

$$\{t(n/m)\}_{n,m=1}^{\infty}, \{h(nm)\}_{n,m=1}^{\infty} \text{ in } \ell^2(\mathbb{N})$$

$$\text{Symbol: } \varphi(t) = \sum_{q \in \mathbb{Q}_+} \widehat{\varphi}(q) q^{-it}$$

$$T_{\infty}(\varphi)f = P_{\infty}(\varphi \cdot f), H_{\infty}(\varphi)f = P_{\infty}(\varphi \cdot f(-it))$$

$$\langle T_{\infty}(\varphi)n^{-it}, m^{-it} \rangle_{\mathcal{H}^2} = \widehat{\varphi}(m/n)$$

$$\langle H_{\infty}(\varphi)n^{-it}, m^{-it} \rangle_{\mathcal{H}^2} = \widehat{\varphi}(mn)$$

Example: Multipliers on \mathcal{H}^2

$$\varphi \in \mathcal{H}^{\infty}: T_{\infty}(\varphi) = \|\varphi\|_{\mathcal{H}^{\infty}}$$

Toeplitz 1938, Hedenmalm-Lindqvist-Seip 1997

BEMERKUNG. Bedient man sich der Sprechweise der Theorie der unendlich vielen Variablen, so ist $D_n(x, \bar{x})$ nichts anderes als der n -te Abschnitt der Bilinearform von unendlich vielen Variablen

$$\begin{aligned} D(x, y) = & c_1(x_1y_1 + x_2y_2 + \dots) + c_2(x_1y_2 + x_2y_4 + \dots) \\ & + c_3(x_1y_3 + x_2y_6 + \dots) + \dots, \end{aligned}$$

nur dass in den n -ten Abschnitt $D_n(x, y)$ hiervon noch $y_1 = \bar{x}_1, \dots, y_n = \bar{x}_n$ gesetzt ist. Nun besagt ein einfacher Hilfssatz³ über Bilinearformen von endlich vielen Veränderlichen $A(x, y)$, dass das Maximum von $|A(x, y)|$ für $\sum x_a \bar{x}_a = 1, \sum y_a \bar{y}_a = 1$ höchstens gleich dem Doppelten des Maximums von $A(x, \bar{x})$ für $\sum x_a \bar{x}_a = 1$ ist; also liegt im vorliegenden Falle $|D_n(x, y)|$ unter $2M$, und Theorem I kann auch so formuliert werden:

Befriedigt eine Belegung die eingangs dieses Paragraphen aufgeführten Voraussetzungen, so soll die aus ihren Hadamardschen Koeffizienten c_n aufgebaute Bilinearform $D(x, y)$ als die zu der Belegung gehörige „ D -Form“ bezeichnet werden. Ist die Belegung beschränkt, $|f(t_i)| \leq M$, so ist die D -Form im Hilbertschen Sinne beschränkt, und ihre obere Schranke liegt unter $2M$.

Die Koeffizientenmatrix einer solchen D -Form hat die Eigentümlichkeit, dass die Stellen, die den nämlichen Koeffizienten tragen, je eine gerade Linie erfüllen und dass alle diese geraden Linien ein Strahlenbüschel bilden mit endlichem Zentrum:⁴

$$D = \left(\begin{array}{cccccccccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & \cdot & \cdot & \cdot \\ 0 & c_1 & 0 & c_2 & 0 & c_3 & 0 & c_4 & 0 & c_5 & 0 & c_6 & \cdot & \cdot & \cdot \\ 0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & c_2 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right).$$

³ Vgl. eine inzwischen in der *Mathematischen Zeitschrift* (Bd. 2 (1918), S. 187)

Multiplicative Toeplitz and Hankel operators in $H^2(\mathbb{T}^\infty)$

Notation

- $\mathbb{N} \ni n = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} \dots =: p^\alpha$,
 $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{Z}_+^{(\infty)}$
- $\mathbb{N} \simeq \mathbb{Z}_+^{(\infty)}$, $\ell^2(\mathbb{N}) \simeq \ell^2(\mathbb{Z}_+^{(\infty)})$

Set $n = p^\alpha$, $m = p^\beta$:

$$\begin{aligned} \{t(n/m)\}_{n,m \in \mathbb{N}} &\simeq \{t(p^{\alpha-\beta})\}_{\alpha,\beta \in \mathbb{Z}_+^{(\infty)}} \\ \{h(nm)\}_{n,m \in \mathbb{N}} &\simeq \{h(p^{\alpha+\beta})\}_{\alpha,\beta \in \mathbb{Z}_+^{(\infty)}} \end{aligned}$$

Infinite multi-torus

$\mathbb{T}^\infty = \mathbb{T} \times \mathbb{T} \times \dots$ compact Abelian group,
 \exists invariant measure m_∞

$L^2(\mathbb{T}^\infty)$, ONB: $\{z^\alpha\}_{\alpha \in \mathbb{Z}^{(\infty)}}$

$H^2(\mathbb{T}^\infty) \subset L^2(\mathbb{T}^\infty)$, ONB: $\{z^\alpha\}_{\alpha \in \mathbb{Z}_+^{(\infty)}}$

$P_\infty : L^2(\mathbb{T}^\infty) \rightarrow H^2(\mathbb{T}^\infty)$

Toeplitz and Hankel operators

For a symbol $\varphi \in L^\infty(\mathbb{T}^\infty)$, define operators
in $H^2(\mathbb{T}^\infty)$:

$$T_\infty(\varphi)f = P_\infty(\varphi \cdot f), \quad H_\infty(\varphi)f = P_\infty(\varphi \cdot f(\bar{z}))$$

$$\begin{array}{ccc} \ell^2(\mathbb{N}) & \longrightarrow & \ell^2(\mathbb{Z}_+^{(\infty)}) \\ \downarrow & & \downarrow \\ \mathcal{H}^2 & \longrightarrow & H^2(\mathbb{T}^\infty) \end{array}$$

Bohr's lift

Boundedness and spectral properties

$$\begin{aligned} T_\infty(\varphi)f &= P_\infty(\varphi \cdot f) \\ H_\infty(\varphi)f &= P_\infty(\varphi \cdot f(\bar{z})) \end{aligned} \quad f \in \mathcal{H}^2(\mathbb{T}^\infty)$$

Boundedness

- $\|T_\infty(\varphi)\| = \|\varphi\|_{L^\infty(\mathbb{T}^\infty)}$
- $\|H_\infty(\varphi)\| \leq \|\varphi\|_{L^\infty(\mathbb{T}^\infty)}$, but Nehari's theorem is false!

Ortega-Cerda+Seip constructed a sequence $\{\varphi_n\}_{n=1}^\infty$:

$$\|H_\infty(\varphi_n)\| \leq C, \inf\{\|\psi\|_{L^\infty} : P_\infty\psi = P_\infty\varphi_n\} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

- Helson 2006: $H_\infty(\varphi) \in \mathfrak{S}_2 \Rightarrow \exists$ bounded symbol
- Brevig-Perfekt 2015: false for \mathfrak{S}_p , $p > 5.74$

Other

- Compactness? Schatten classes?
- $H_\infty(\varphi)$ of finite rank: Perfekt-Pushnitski 2018

Work on multiplicative Toeplitz/Hankel operators to date

- General questions: boundedness, Nehari's theorem, etc.
- Arithmetic multiplicative Toeplitz matrices, e.g. $t(n/m) = \frac{(\gcd(n, m))^{2\sigma}}{(nm)^\sigma}$

Often are multiplicative: $t(p^\alpha) = \prod_{p_i} t(p_i^{\alpha_i})$

Multiplicativity leads to tensor product representation $\otimes_p \{t(p^{j-k})\}_{j,k \in \mathbb{Z}_+}$ (Hilberdink)

- Multiplicative Hankel matrices: restrictions $h(nm)$ where $h(xy)$ is a “nice” kernel

Comparision with integral operators $f \mapsto \int_1^\infty h(xy)f(y)dy$

See Naz Miheisi's talk

The multiplicative Hilbert matrix

- $\{H_\infty(\varphi)\} = \{(nm)^{-\frac{1}{2}}(\log nm)^{-1}\}_{n,m=2}^\infty$ $h(nm) \rightarrow \{h(nm)\}$, $h(n) = n^{-\frac{1}{2}}(\log n)^{-1}$
- Brevig-Perfekt-Seip-Siskakis-Vukotic 2016, Perfekt-Pushnitski 2018
- Heuristics: related to $f \mapsto \int_2^\infty (st)^{-\frac{1}{2}}(\log st)^{-1} f(s) ds$ in $L^2(2, \infty)$ $s=e^x, t=e^y$
- Change of variable: $F \mapsto \int_{\log 2}^\infty (x+y)^{-1} F(y) dy$ in $L^2(\log 2, \infty)$
- “1/2 of the Carleman operator” $\text{Carleman: } \frac{1}{x+y} \text{ in } L^2(0, \infty) \quad \sigma = [0, \pi], \text{mult}=2$
- $H_\infty(\varphi)$ has purely a.c. spectrum $[0, \pi]$, multiplicity=1.
- No explicit diagonalisation known!
- $H_\infty(\varphi)$ is unitarily equivalent to the integral operator in $L^2(0, \infty)$ with the Hankel kernel $\zeta(x+y+1) - 1$ $H_\infty(\varphi) = L^* L ; \quad L L^*$
- Dirichlet symbol: $\varphi(t) = \sum_{n=2}^\infty (\log n)^{-1} n^{-\frac{1}{2}-it}, \varphi'(t) = -i\zeta(\frac{1}{2} + it)$
- Does it have a bounded symbol?

Classical Hilbert matrix:

$$h(j) \rightarrow \{h(j+k)\} = H$$

$$\|h(j)\| = O\left(\frac{1}{j}\right) \Rightarrow H - \text{bdd}$$

$$h(j) = o\left(\frac{1}{j}\right) \Rightarrow H - \text{compact}$$

$h(j)$ slower than $1/j$ $\Rightarrow H$ is unbdd

$$h(st) \quad f \mapsto \int_1^\infty h(st) f(s) ds \in L^2(1, \infty)$$

$$\{f_n\} \mapsto \sum_{m=1}^\infty h(nm) f_m$$

$$f \mapsto \int_0^\infty \frac{1}{x+y} f(y) dy \quad \text{in } L^2(0, \infty) \quad [0, \pi] \text{ mult=2}$$

$$\left\{ \frac{1}{n+m} \right\}_{n,m=1}^\infty \quad [0, \infty] \text{ mult=1}$$

$$\uparrow$$

$$\int_1^\infty \frac{1}{x+y} f(y) dy \quad \text{in } L^2(1, \infty), [0, \bar{n}], \text{ mult=1}$$

Truncations of multiplicative Toeplitz operators I: Følner sequences

- $\varphi \in L^\infty(\mathbb{T}^\infty)$, consider $T_\infty(\varphi)$ in $H^2(\mathbb{T}^\infty)$
- $\omega \subset \mathbb{N}$: $\{T_\infty(\varphi)\}_\omega = \{\widehat{\varphi}(n/m)\}_{n,m \in \omega}$
- Szegő type theorem: $\omega_1 \subset \omega_2 \subset \dots \subset \mathbb{N}$, $\varphi = \widehat{\varphi}$

$$\lim_{N \rightarrow \infty} \frac{\#\{n : \lambda_n(\{T_\infty(\varphi)\}_{\omega_N}) \in \Delta\}}{\#\omega_N} = m_\infty\{z \in \mathbb{T}^\infty : \varphi(z) \in \Delta\}$$

if $\{\omega_N\}$ is a multiplicative Følner sequence:

$$\lim_{\#\omega_N \rightarrow \infty} \frac{\#\{k \in \omega_N : nk \in \omega_N\}}{\#\omega_N} = 1, \quad \forall n \in \mathbb{N}$$

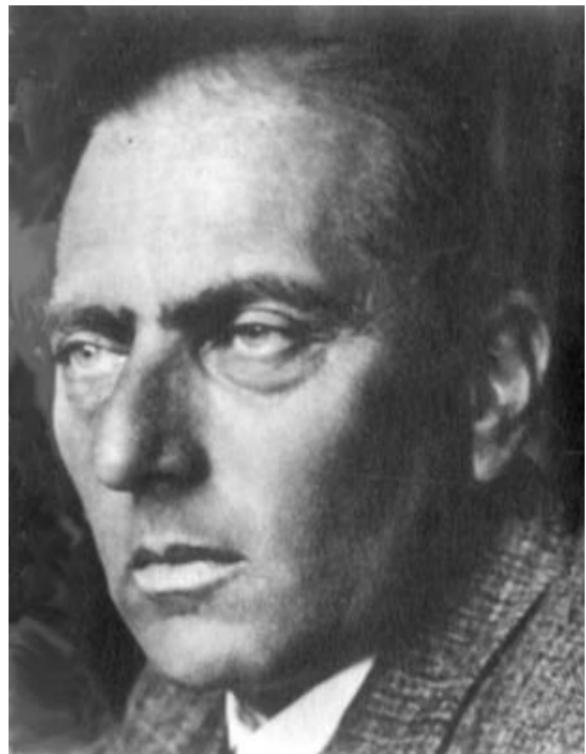
- Bédos 1997, Nikolski+Pushnitski 2020
- $\omega_N = \{1, \dots, N\}$ is NOT multiplicative Følner!
- A multiplicative Følner sequence:

$$\omega_N = \{p^\alpha : 0 \leq \alpha_j \leq A_j^{(N)}\}, \quad \forall j: A_j^{(N)} \rightarrow \infty \text{ as } N \rightarrow \infty$$

Truncations of multiplicative Toeplitz operators II: truncations to $\{1, \dots, N\}$

- $\{T_\infty(\varphi)\}_N = \{\widehat{\varphi}(n/m)\}_{n,m=1}^N$
- Fix $\sigma > 0$, let $\varphi_\sigma(t) = |\zeta(\sigma + it)|^2$; consider $T_\infty(\varphi_\sigma)$.
- $[T_\infty(\varphi_\sigma)]_{n,m} = \zeta(2\sigma) \frac{(\gcd(n, m))^{2\sigma}}{(nm)^\sigma}, \sigma > 1$
- For $\sigma \leq 1$, $|\zeta(\sigma + it)|^2$ is unbounded, and so is $T_\infty(\varphi_\sigma)$.
- What about the truncation $\{T_\infty(\varphi_\sigma)\}_N$?
Is it related to $|\zeta(\sigma + it)|^2$ for $|t| \leq T = T(N)$?
- Norms and singular values of $\{T_\infty(\varphi_\sigma)\}_N$ as $N \rightarrow \infty$:
Lindqvist-Seip 1998, Hilberdink 2009, Bondarenko-Hilberdink-Seip 2016, Aistleitner 2016
- Sample result for $\frac{1}{2} \leq \sigma \leq 1$:

$$\sup_{|t| \leq T} |\zeta(\sigma + it)|^2 \geq \|\{T_\infty(\varphi_\sigma)\}_N\| + \text{small error}, \quad N = T^{\frac{2}{3}(\sigma - \frac{1}{2}) - \varepsilon}$$



Otto Toeplitz (1881-1940)



Hermann Hankel (1839-1873)