Hardy spaces of general Dirichlet series and their maximal inequalities

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$\boldsymbol{\lambda}$-Dirichlet series

A natural space of general Dirichlet series

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Definition

Let $\mathcal{D}_\infty(\lambda)$ denote the space of all λ-Dirichlet series $\sum a_n e^{-\lambda_n s}$ that **converge on** $[\mathbf{Re} > \mathbf{0}]$ and define a **bounded limit function**

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Theorem (Defant + S. [11], 2020)

For every frequency $\lambda = (\lambda_n)$ *and* $D \in \mathcal{D}_\infty(\lambda)$

$$\sup_{n \in \mathbb{N}} |a_n(D)| \leq \|D\|_\infty.$$

In particular, $(\mathcal{D}_\infty(\lambda), \|\cdot\|_\infty)$ *defines a* **normed space**.

Spaces of general Dirichlet series

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In general $(\mathcal{D}_\infty(\lambda), \|\cdot\|_\infty)$ **does not form a Banach space.**

$\exists$ 'natural' Banach space $\mathcal{X} \subsetneq H_\infty[Re > 0]$: $\mathcal{D}_\infty(\lambda) \hookrightarrow \mathcal{X}$ and

$\mathcal{D}_\infty(\lambda)$ is a Banach space $\Leftrightarrow$ $\mathcal{X} = \mathcal{D}_\infty(\lambda)$

Holomorphic almost periodic functions on half planes

Definition

Let $\mathcal{H}_\infty^\lambda[Re > 0]$ denote the space of all **holomorphic and bounded** $g\colon [Re > 0] \to \mathbb{C}$ such that every restriction

$$g_\sigma\colon \mathbb{R} \to \mathbb{C}, \quad t \mapsto g(\sigma + it), \ \sigma > 0$$

defines an **almost periodic function** on $\mathbb{R}$ and for every $x \in \mathbb{R}$ the xth Bohr coefficient of g

$$a_x(g) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g_\sigma(it) e^{(\sigma+it)x} dt$$

vanishes, whenever $x \notin \{\lambda_n \mid n \in \mathbb{N}\}$.

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Theorem

$\mathcal{H}_\infty^\lambda[Re > 0]$ *forms a* **Banach space** *for every frequency* λ.

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Let λ *be an* **arbitrary** *frequency. Then isometrically*

$$\mathcal{D}_\infty(\lambda) \hookrightarrow \mathcal{H}_\infty^\lambda[Re > 0], \quad D \mapsto g.$$

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Let λ be an **arbitrary** *frequency. Then*

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Crucial part "$\Rightarrow$": Assuming $\mathcal{D}_\infty(\lambda)$ is complete, let $g \in \mathcal{H}_\infty^\lambda[Re > 0]$ with Dirichlet series $D = \sum a_{\lambda_n}(g)e^{-\lambda_n s}$.

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Linking element: Bohr's theorem

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Definition

We say that **Bohr's theorem holds for $\boldsymbol{\lambda}$**, whenever every somewhere convergent λ-Dirichlet series D with limit function g, that extents to $[Re > 0]$ to a holomorphic and bounded function, converges uniformly on $[Re > \varepsilon]$, $\varepsilon > 0$.

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Let $\mathcal{D}_{\infty}^{ext}(\lambda)$ denote the space of all **somewhere convergent** $D = \sum a_n e^{-\lambda_n s}$ whose limit function extent to $[Re > 0]$ to a bounded and holomorphic function.

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Theorem

For every frequency λ we isometrically have

$$\mathcal{D}_{\infty}(\lambda) \subset \mathcal{D}_{\infty}^{ext}(\lambda) \subset \mathcal{H}_{\infty}^{\lambda}[Re > 0].$$

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Observation: Bohr's theorem holds for λ, whenever every $D \in \mathcal{D}_\infty^{ext}(\lambda)$ converges uniformly on $[Re > \varepsilon]$, $\varepsilon > 0$.

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$$L(\lambda) := \sup_{D \in \mathcal{D}(\lambda)} \sigma_a(D) - \sigma_c(D) = 0 \text{ (Bohr, 1913)}$$

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Theorem (Defant + S. [5], 2020)

Let λ be a frequency. Then TFAE:

(1) *Bohr's theorem holds for λ.*

(2) $\mathcal{D}_\infty(\lambda)$ *is a Banach space.*

(3) $\mathcal{D}_\infty(\lambda) = \mathcal{H}_\infty^\lambda[Re > 0]$.

(4) *For every $\sigma > 0$ there exists $C = C(\lambda, \sigma)$ such that*

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Crucial ingredients of the proof:

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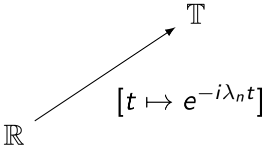

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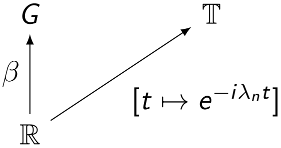

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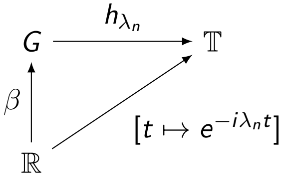

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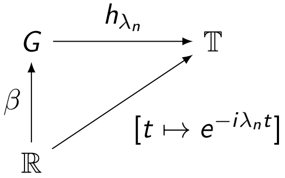

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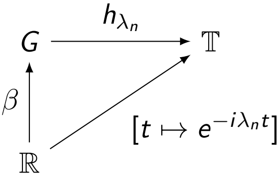

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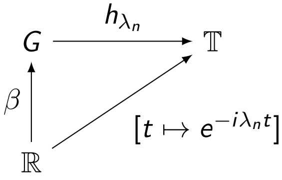

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! For every frequency $\boldsymbol{\lambda}$ there is such a pair (G,β) !

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Theorem (Defant + S. [7], 2020)

! $\mathcal{H}_p(\lambda)$ *is* **independent** *of the chosen λ-Dirichlet group* **!**

Examples

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$$\overline{\mathbb{R}} = \{\gamma\colon (\mathbb{R},+) \to \mathbb{T} \mid \gamma \text{ homomorphism}\},$$

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Bayart's invention of $\mathcal{H}_p$-spaces, 2002:

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(2) Links between $\mathcal{D}_\infty(\lambda)$, $\mathcal{H}_\infty(\lambda) = H_\infty^\lambda(G)$ and $H_\infty^\lambda[Re > 0]$?

Special case $p = \infty$

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Theorem

Let $\lambda = (\lambda_n)$ be an **arbitrary frequency** *and (G, β) a λ-Dirichlet group. Then as Banach spaces*

$$\boldsymbol{H_\infty^\lambda(G) = \mathcal{H}_\infty(\lambda) = \mathcal{H}_\infty^\lambda[Re > 0]},$$

such that

$$\widehat{f}(h_{\lambda_n}) = a_n(D) = a_{\lambda_n}(g).$$

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Special case $p = 2$: The Carleson-Hunt theorem

Theorem (see [5] and [3])

Let $\boldsymbol{\lambda}$ be arbitrary *with λ-Dirichlet group (G, β). Then for every* $\boldsymbol{f \in H_2^\lambda(G)}$

$$\Big\|\omega \mapsto \sup_N \Big|\sum_{n=1}^{N} \widehat{f}(h_{\lambda_n})h_{\lambda_n}(\omega)\Big|\Big\|_2 \leq CH_2\|f\|_2.$$

In particular, **almost everywhere** *on G*

$$\boldsymbol{f = \sum_{n=1}^{\infty} \widehat{f}(h_{\lambda_n})h_{\lambda_n}}$$

Proof of Bohr's theorem implies completeness

Corollary

Let $D = \sum a_n e^{-\lambda_n s}$ with $\mathbf{(a_n) \in \ell_2}$ *and $f \in H_2^\lambda(G)$ such that $a_n = \widehat{f}(h_{\lambda_n})$ for all $n \in \mathbb{N}$. Then for* **almost every** $\boldsymbol{\omega \in G}$ *the vertical limit of D*

$$\boldsymbol{D^\omega(s) = \sum a_n h_{\lambda_n}(\omega) e^{-\lambda_n s}}$$

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Hence

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Lemma

Let $\lambda = (\lambda_n)$ be **arbitrary** *with λ-Dirichlet group (G, β) and $g \in \mathcal{H}_\infty^\lambda[Re > 0]$. Then with $a_n = a_{\log n}$, $n \in \mathbb{N}$*

$$\exists\, \omega \in G: \sum a_n h_{\lambda_n}(\omega) e^{-\lambda_n s} \in \mathcal{D}_\infty(\lambda).$$

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Together,

$$D \text{ converges uniformly on } [Re > \varepsilon], \varepsilon > 0.$$

The full equivalence theorem

Theorem

(1) *Bohr's theorem holds for λ.*

(2) $\mathcal{D}_\infty(\lambda)$ *is a Banach space.*

(3) $\mathcal{D}_\infty(\lambda) = \mathcal{H}_\infty^\lambda[Re > 0]$.

(4) *For every $\sigma > 0$ there exists $C = C(\lambda, \sigma)$ such that*

$$\sup_N \| \sum_{n=1}^{N} a_n(D) e^{-\sigma\lambda_n} e^{-\lambda_n s} \|_\infty \leq C \|D\|_\infty, \ D \in \mathcal{D}_\infty(\lambda).$$

(5) $\mathcal{D}_\infty(\lambda) = \mathcal{H}_\infty(\lambda)$.

(6) *Bayart's Montel theorem holds in $\mathcal{D}_\infty(\lambda)$.*

Immediately arising (vague) questions

(1) Let $f \in H_p^\lambda(G)$, where $1 \leq p \leq \infty$. What can we say about **convergence respectively summability of the Fourier series** of f

$$\sum \widehat{f}(h_{\lambda_n})h_{\lambda_n} \ ?$$

(2) Links between $\mathcal{D}_\infty(\lambda)$, $\mathcal{H}_\infty(\lambda) = H_\infty^\lambda(G)$ and $H_\infty^\lambda[Re > 0]$?

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Reflexive case $1 < p < \infty$

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Theorem (Defant + S. [5], 2020)

Let λ be arbitrary *with λ-Dirichlet group (G, β) and* $\boldsymbol{1 < p < \infty}$. *Then for every* $\boldsymbol{f \in H_p^\lambda(G)}$

$$\Big\|\omega \mapsto \sup_N \Big|\sum_{n=1}^{N} \widehat{f}(h_{\lambda_n}) h_{\lambda_n}(\omega)\Big|\Big\|_p \leq CH_p \|f\|_p.$$

In particular, **almost everywhere** *on G*

$$\boldsymbol{f = \sum_{n=1}^{\infty} \widehat{f}(h_{\lambda_n}) h_{\lambda_n}}$$

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If λ satisfies **(LC)**, *then for every λ-Dirichlet group (G, β) for every $u > 0$ there exists $\exists C > 0$ such that for every $f \in H_1^\lambda(G)$*

$$\|\sup_{\sigma > u} \sup_N |\sum_{n=1}^{N} \widehat{f}(h_{\lambda_n}) e^{-\sigma\lambda_n} h_{\lambda_n}|\|_{1,\infty} \leq C\|f\|_1.$$

In particular, for almost every $\omega \in G$ for every $u > 0$

$$\boldsymbol{\sum_{n=1}^{\infty} \widehat{f}(h_{\lambda_n}) e^{-u\lambda_n} h_{\lambda_n}(\omega) = f * p_u(\omega).}$$

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Bayart [1], 2021: $\exists$ λ, λ-Dirichlet group (G, β) and $f \in H_1^\lambda(G)$ such that for every $u > 0$

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Let $f \in H_1^\lambda(G)$ and $x, k \geq 0$. Then the polynomial

$$R_x^{\lambda,k}(f) = \sum_{\lambda_n < x} \widehat{f}(h_{\lambda_n})\big(1 - \frac{\lambda_n}{x}\big)^k h_{\lambda_n}$$

is called the (λ, k)-**Riesz mean** of f of length x and order k.

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For every $k > 0$ there is a constant $C = C(k)$ such that for every frequency λ and $\boldsymbol{f} \in \boldsymbol{H_1^\lambda(G)}$ we have

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$$\lambda = (\lambda_n) \text{ arbitrary: Principle of localization? Dini-test?}$$

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