On C^*-algebras associated to product systems and semi-saturated Fell bundles

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Correspondences

Let A and B be C^*-algebras.

- A *correspondence* from A to B is a Hilbert B-module $\mathcal{F}$ with a left action of A implemented by a *nondegenerate* *-homomorphism $\varphi\colon A \to \mathbb{B}(\mathcal{F})$. It is *proper* if $\varphi(A) \subseteq \mathbb{K}(\mathcal{F})$. It is *faithful* if φ is injective.
- A *Hilbert A, B-bimodule* is a (right) Hilbert B-module $\mathcal{F}$ with a *left* Hilbert A-module structure $\langle\langle \cdot \mid \cdot \rangle\rangle_A$ such that $\langle\langle \xi \mid \eta \rangle\rangle_A \zeta = \xi \langle \eta \mid \zeta \rangle_B$ for all $\xi, \eta, \zeta \in \mathcal{F}$.

Let $\mathcal{E}\colon A \rightsquigarrow A$ be a correspondence. A *representation* of $\mathcal{E}$ in a C^*-algebra B is a pair (π, t), where $\pi\colon A \to B$ is a *-homomorphism, $t\colon \mathcal{E} \to B$ is a linear map, and

1. $\pi(a)t(\xi) = t(\varphi(a)\xi)$ for all $a \in A$ and $\xi \in \mathcal{E}$;
2. $t(\xi)^* t(\eta) = \varphi(\langle \xi \mid \eta \rangle_A)$ for all $\xi, \eta \in \mathcal{E}$.

These conditions imply $t(\xi)\pi(a) = t(\xi a)$ for all $\xi \in \mathcal{E}$ and $a \in A$.

Relative Cuntz–Pimsner algebras

For any representation (π, t) of $\mathcal{E}$ in B, there is a unique *-homomorphism $\pi^1\colon \mathbb{K}(\mathcal{E}) \to B$ with $\pi^1(|\xi\rangle\langle\eta|) = t_\xi t_\eta^*$ for all $\xi, \eta \in \mathcal{E}$ **(Pimsner, 1997)**.

Let J be any ideal of A with $\varphi(J) \subseteq \mathbb{K}(\mathcal{E})$.

- The *relative Cuntz–Pimsner algebra* $\mathcal{O}_{J,\mathcal{E}}$ of $\mathcal{E}$ is the C^*-algebra generated by A and $\mathcal{E}$ which is universal for representations of $\mathcal{E}$ *covariant* on J in the sense that

$$\pi(a) = \pi^1(\varphi(a)) \qquad \text{for all } a \in J$$

  (Pimsner, 1997; Muhly–Solel, 1998).
- $I_\mathcal{E} = \varphi^{-1}(\mathbb{K}(\mathcal{E})) \cap (\ker\varphi)^\perp \Rightarrow$
  the representation of $\mathcal{E}$ in $\mathcal{O}_\mathcal{E}$ is faithful $+$
  gauge-invariant uniqueness theorem **(Katsura, 2004)** .

Fell bundle associated to $\mathcal{O}_{J,\mathcal{E}}$

- $\mathcal{O}_{J,\mathcal{E}}$ carries a *$\mathbb{Z}$-grading* arising from a canonical gauge action of $\mathbb{T}$ on $\mathcal{O}_{J,\mathcal{E}}$;
- the associated Fell bundle $\{\mathcal{O}^n_{J,\mathcal{E}}\}_{n\in\mathbb{Z}}$ is *semi-saturated*: $\mathcal{O}^m_{J,\mathcal{E}} \cdot \mathcal{O}^n_{J,\mathcal{E}} = \mathcal{O}^{m+n}_{J,\mathcal{E}}$ if $m, n \geq 0$ **(Exel, 1994)**.
- $\mathcal{E}$ Hilbert A-bimodule and $J = I_{\mathcal{E}} = \langle\langle \mathcal{E} \mid \mathcal{E} \rangle\rangle \Rightarrow \mathcal{O}^n_{I_{\mathcal{E}},\mathcal{E}} \cong \mathcal{E}^{\otimes n}$.
- Semi-saturated Fell bundles with unit fibre $A \leftrightarrow$ Hilbert A-bimodules.
- Relative Cuntz–Pimsner algebras of Hilbert bimodules determined by Katsura's ideal $\leftrightarrow$ Cross sectional C^*-algebras of semi-saturated Fell bundles $\leftrightarrow$ crossed products by Hilbert bimodules $A \rtimes_{\mathcal{E}} \mathbb{Z}$, in the sense of **(Abadie–Eilers–Exel, 1998)**.

Product systems

(Fowler, 2002) Let P be a semigroup with identity e and A be a C^*-algebra. A *product system* over P of A-correspondences consists of:

1. a correspondence $\mathcal{E}_p \colon A \rightsquigarrow A$ for each $p \in P$;
2. correspondence isomorphisms $\mu_{p,q} \colon \mathcal{E}_p \otimes_A \mathcal{E}_q \xrightarrow{\cong} \mathcal{E}_{pq}$ for all $p, q \in P$,

where $\mathcal{E}_e =_A A_A$ and the multiplication maps $\mu_{e,p}$ and $\mu_{p,e}$ are required to satisfy $\mu_{e,p}(a \otimes \xi_p) = \varphi_p(a)\xi_p$ and $\mu_{p,e}(\xi_p \otimes a) = \xi_p a$ for all $a \in A$ and $\xi_p \in \mathcal{E}_p$. The multiplication maps are also assumed to be associative. If each $\mathcal{E}_p$ is a Hilbert A-bimodule, we will speak of a *product system of Hilbert bimodules*.

Example: A unital C^*-algebra, $\alpha \colon P \to \mathrm{End}(A)$ semigroup homomorphism with $\alpha_e = \mathrm{id}_A$ gives a (proper) product system over P^{op}: $\mathcal{E}_p = \alpha_p(1)A$, $\mu_{p,q}(\alpha_p(1)a \otimes_A \alpha_q(1)b) = \alpha_{qp}(1)\alpha_q(a)b$.

Representations of product systems

A *Toeplitz representation* of $\mathcal{E} = (\mathcal{E}_p)_{p\in P}$ in a C^*-algebra B consists of linear maps $t_p\colon \mathcal{E}_p \to B$, for all $p \in P \setminus \{e\}$, and a *-homomorphism $t_e\colon A \to B$, satisfying the following two axioms:

(T1) $t_p(\xi)t_q(\eta) = t_{pq}(\mu_{p,q}(\xi \otimes_A \eta))$ for all $p, q \in P$, $\xi \in \mathcal{E}_p$ and $\eta \in \mathcal{E}_q$;

(T2) $t_p(\xi)^* t_p(\eta) = t_e(\langle \xi \,|\, \eta \rangle)$, for all $p \in P$ and $\xi, \eta \in \mathcal{E}_p$.

Fock representation: Let P be a left cancellative semigroup and let $\mathcal{E}^+ = \bigoplus_{p\in P} \mathcal{E}_p$ be a product system. There is a representation of $\mathcal{E}$ in $\mathbb{B}(\mathcal{E}^+)$ such that

$$t_p(\xi_p)(\eta_q) = \mu_{p,q}(\xi_p \otimes \eta_q) \in \mathcal{E}_{pq}$$

for all $p, q \in P$, $\xi_p \in \mathcal{E}_q$ and $\eta_q \in \mathcal{E}_q$.

Quasi-lattice ordered groups and compactly aligned product systems

Let G be a group and P a subsemigroup of G.

- (G, P) is a *quasi-lattice ordered group* if $P \cap P^{-1} = \{e\}$ and given elements g_1, g_2 of G that have a common upper bound in P with respect to the partial order $g_1 \leq g_2 \Leftrightarrow g_1^{-1} g_2 \in P$, there is a least upper bound $g_1 \vee g_2$ in P **(Nica, 1992)**. Examples: $(\mathbb{Z}^k, \mathbb{N}^k)$, $(\mathbb{F}, \mathbb{F}^+) \ldots$
- Given a product system, there is a *-homomorphism $\iota_p^{pq} \colon \mathbb{B}(\mathcal{E}_p) \to \mathbb{B}(\mathcal{E}_{pq})$: ι_p^{pq} sends $T \in \mathbb{B}(\mathcal{E}_p)$ to $\mu_{p,q} \circ T \circ \mu_{p,q}^{-1}$.
- **(Fowler, 2002)** If (G, P) is a quasi-lattice ordered group, we say that $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ is *compactly aligned* if, for all $p, q \in P$ with $p \vee q < \infty$, we have

$$\iota_p^{p \vee q}(T) \iota_q^{p \vee q}(S) \in \mathbb{K}(\mathcal{E}_{p \vee q}), \qquad \text{for all } T \in \mathbb{K}(\mathcal{E}_p) \text{ and } S \in \mathbb{K}(\mathcal{E}_q).$$

Nica covariant representations

- **(Nica, 1992);(Fowler, 2002)** A representation $t = \{t_p\}_{p\in P}$ of $\mathcal{E}$ in a C^*-algebra B is *Nica covariant* if for all $p, q \in P$, $T \in \mathbb{K}(\mathcal{E}_p)$ and $S \in \mathbb{K}(\mathcal{E}_q)$, we have

$$t^{(p)}(T)t^{(q)}(S) = \begin{cases} t^{(p\vee q)}\big(\iota_p^{p\vee q}(T)\iota_q^{p\vee q}(S)\big) & \text{if } p\vee q < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

- If t is Nica covariant, then $\mathrm{C}^*(t(\mathcal{E}))$ is spanned by

$$\{t_p(\xi_p)t_q(\eta_q)^* \mid p, q \in P, \xi_p \in \mathcal{E}_p, \eta_q \in \mathcal{E}_q\}.$$

- Fock representation: Let $\mathcal{E} = (\mathbb{C}\delta_p)_{p\in P}$ and $\mathcal{E}^+ = \bigoplus_{p\in P} \mathbb{C}\delta_p$. Let $v_p \in \mathbb{B}(\mathcal{E}^+)$ be the isometry given by $v_p(\delta_q) = \delta_{pq}$. Then v is Nica covariant:

$$v_p v_p^* v_q v_q^* = \begin{cases} v_{(p\vee q)} v_{(p\vee q)}^* & \text{if } p\vee q < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Simplifiable product systems of Hilbert bimodules

A product system of *Hilbert bimodules* $\mathcal{E} = (\mathcal{E}_p)_{p\in P}$ will be called *simplifiable* if for all $p, q \in P$ one has

(i) $\langle\langle \mathcal{E}_p \,|\, \mathcal{E}_p \rangle\rangle \langle\langle \mathcal{E}_q \,|\, \mathcal{E}_q \rangle\rangle \subseteq \langle\langle \mathcal{E}_{p\vee q} \,|\, \mathcal{E}_{p\vee q} \rangle\rangle \quad$ if $p \vee q < \infty$;

(ii) $\langle\langle \mathcal{E}_p \,|\, \mathcal{E}_p \rangle\rangle \langle\langle \mathcal{E}_q \,|\, \mathcal{E}_q \rangle\rangle = \{0\} \quad$ if $p \vee q = \infty$,

where $\langle\langle \cdot \,|\, \cdot \rangle\rangle$ denotes the left A-valued inner-product.

- Simplifiable $\Rightarrow$ compactly aligned.
- Compactly aligned $\not\Rightarrow$ simplifiable.
- Example: Let A be a unital C^*-algebra and let $\alpha\colon P \to \mathrm{End}(A)$ be an action by injective endomorphisms with hereditary range. Assume that P is $Pp \cap Pq \neq \emptyset$ for all $p, q \in P$. Then the product system over P^{op} given by $\mathcal{E}_p = \alpha_p(1)A$ is a simplifiable product system of Hilbert bimodules with

$$\langle\langle \alpha_p(1)a \,|\, \alpha_p(1)b \rangle\rangle \coloneqq \alpha_p^{-1}(\alpha_p(1)ab^*\alpha_p(1)) \quad \textbf{(Larsen, 2010)}.$$

Fell bundle associated to a simplifiable product system

Proposition (Meyer, S.,2017)

Let (G, P) be a quasi-lattice ordered group and $\mathcal{E} = (\mathcal{E}_p)_{p\in P}$ a simplifiable product system of Hilbert bimodules. Then $\mathcal{E}$ extends to a Fell bundle over G.

Idea of the proof: We set

$$B_g := \begin{cases} \mathcal{E}_{g\vee e} \otimes_A \mathcal{E}^*_{g^{-1}\vee e} & \text{if } g \vee e < \infty, \\ \{0\} & \text{otherwise.} \end{cases}$$

So $B_p \cong \mathcal{E}_p$ for all $p \in P$ and

$$\begin{aligned} \mathcal{E}_p \otimes_A \mathcal{E}^*_q \otimes_A \mathcal{E}_r \otimes_A \mathcal{E}^*_s &\cong \mathcal{E}_p \otimes_A \mathcal{E}^*_q \otimes_A \left(\mathcal{E}_q \otimes_A \mathcal{E}^*_q \otimes_A \mathcal{E}_r \otimes_A \mathcal{E}^*_r\right) \otimes_A \mathcal{E}_r \otimes_A \mathcal{E}^*_s \\ &\hookrightarrow \mathcal{E}_p \otimes_A \mathcal{E}^*_q \otimes_A \left(\mathcal{E}_{(q\vee r)} \otimes_A \mathcal{E}^*_{(q\vee r)}\right) \otimes_A \mathcal{E}_r \otimes_A \mathcal{E}^*_s \\ &\hookrightarrow \mathcal{E}_{pq^{-1}(q\vee r)}\mathcal{E}^*_{sr^{-1}(q\vee r)}. \end{aligned}$$

Semi-saturated Fell bundles

Let (G, P) be a quasi-lattice ordered group.
We will say that a Fell bundle $\mathcal{B} = (B_g)_{g \in G}$ is *semi-saturated* if satisfies the following conditions:

(S1) $B_p B_q = B_{pq}$ for all $p, q \in P$;

(S2) $B_g = B_{(g \vee e)} B^*_{(g^{-1} \vee e)}$ for all $g \in G$ with $g \vee e < \infty$.

(S3) $B_g = \{0\}$ for all $g \in G$ with $g \vee e = \infty$.

- **(Exel, 2000)** Let $\mathbb{F}$ denote the free group on a set of generators S. A Fell bundle $\mathcal{B} = (B_g)_{g \in \mathbb{F}}$ is *semi-saturated* if $B_p B_q = B_{pq}$ for all $p, q \in \mathbb{F}$ such that the multiplication $p \cdot q$ involves no cancellation. It is called *orthogonal* if $B^*_p B_q = \{0\}$ whenever p and q are distinct generators of $\mathbb{F}$.
- $\mathcal{B} = (B_g)_{g \in \mathbb{F}}$ is semi-saturated and orthogonal in the sense of **(Exel, 2000)** $\Leftrightarrow$ $\mathcal{B} = (B_g)_{g \in \mathbb{F}}$ is semi-saturated with respect to $(\mathbb{F}, \mathbb{F}^+)$.

Simplifiable product systems out of semi-saturated Fell bundles

Proposition (Meyer, S., 2017)

Let $(B_g)_{g\in G}$ be a semi-saturated Fell bundle with respect to (G, P). Then the associated product system $\mathcal{E} = (B_p)_{p\in P}$ is simplifiable.

Given a simplifiable product system $\mathcal{E} = (\mathcal{E}_p)_{p\in P}$, let $\mathcal{I} = \{I_{\mathcal{E}_p}\}_{p\in P}$ be the family of Katsura's ideals for $\mathcal{E}$: that is, $I_{\mathcal{E}_p} = \langle\langle \mathcal{E}_p \,|\, \mathcal{E}_p \rangle\rangle$ for all p in P.

Corollary (Meyer, S., 2017)

Let $(B_g)_{g\in G}$ be a semi-saturated Fell bundle with respect to (G, P). Then its cross sectional C^*-algebra is isomorphic to the relative Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{E},\mathcal{I}}$ of the associated product system of Hilbert bimodules.

Correspondences between semi-saturated Fell bundles

- Let $(B_g)_{g \in G}$ and $(C_g)_{g \in G}$ be semi-saturated Fell bundles. Let $\mathcal{F} \colon B_e \rightsquigarrow C_e$ be a proper correspondence and let $V = \{V_g\}_{g \in G}$ be a family of isometries, commuting with the multiplication maps, so that

$$V_g \colon B_g \otimes_{B_e} \mathcal{F} \to \mathcal{F} \otimes_{C_e} C_g.$$

Suppose that V_p is unitary for all $p \in P$. Then $(\mathcal{F}, \{V_p\}_{p \in P})$ gives rise to a *proper covariant correspondence* from $(B_p)_{p \in P}$ to $(C_p)_{p \in P}$: that is, $V_p \colon B_p \otimes_{B_e} \mathcal{F} \cong \mathcal{F} \otimes_{C_e} C_p$ is a correspondence isomorphism for all $p \in P$ and $\langle\langle B_p \mid B_p \rangle\rangle \mathcal{F} \subseteq F \langle\langle C_p \mid C_p \rangle\rangle$.

Correspondences between semi-saturated Fell bundles

- Let $(B_p)_{p\in P}$ and $(C_p)_{p\in P}$ be simplifiable product systems of Hilbert bimodules and let $(\mathcal{F},\{V_p\}_{p\in P})$ be a *proper covariant correspondence* from $(B_p)_{p\in P}$ to $(C_p)_{p\in P}$. Then $(\mathcal{F},\{V_p\}_{p\in P})$ extends to a correspondence between the associated semi-saturated Fell bundles.

Thank you for your attention!