The tight C*-algebra of an Inverse Semigroup

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Inverse semigroup

A semigroup S is called an inverse semigroup if for every element $s \in S$ there is a unique element s^* such that

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*$$

$E(S)$ = set of idempotents, that is, elements e such that $e^2 = e$.

- Idempotents are self-inverse ($e^* = e$)
- If $e, f \in E(S)$, then $ef \in E(S)$ and $ef = fe$
- For $s \in S$, s^*s and ss^* are idempotent
- $(s^*)^* = s$
- $(st)^* = t^*s^*$

Often we assume that S has a unit (then it's an inverse monoid).

More often we assume that S has a zero ($0s = s0 = 0 \forall s$)

Examples

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- G group, then for each $g \in G$, $g^* = g^{-1}$.
  An inverse semigroup S is a group if and only if $E(S)$ contains only one element.

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- G group, then for each $g \in G$, $g^* = g^{-1}$.
  An inverse semigroup S is a group if and only if $E(S)$ contains only one element.
- Let X be a set, and let

$$\mathcal{I}(X) = \{f : U \to V \mid U, V \subset X, f \text{ bijective}\}$$

Operation: composition (on largest possible set)
Inverse: function inverse
Theorem (Wagner-Preston) every inverse semigroup embeds in some $\mathcal{I}(X)$.

Examples

- 2×2 rook matrices

$$\mathcal{I}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

  This is closed under multiplication, and with $* =$ matrix transpose, is an inverse semigroup.

  Isomorphic to $\mathcal{I}(\{0, 1\})$

- If A is a C*-algebra, $S \subset A$ set of partial isometries closed under product and $*$ is an inverse semigroup.

  $E(S)$ is a commuting set of projections.

Order Structure

An important aspect of an inverse semigroup is the order structure induced by the multiplication.

$$s \leq t \quad \Longleftrightarrow \quad te = s \quad \text{for some } e \in E(S)$$

$E(S)$ is a semilattice, with $e \wedge f = ef$

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On $\mathcal{I}(X)$, $f \leq g$ iff g extends f as a function.

If S is an inverse semigroup of partial isometries, the order on $E(S)$ coincides with usual ordering.

Inverse semigroups in C*-algebras

Question: can every inverse semigroup be realized as a set of partial isometries in some C*-algebra?

Answer: Yes – Paterson (99)

$\pi : S \to A$ is a representation if $\pi(st) = \pi(s)\pi(t)$, $\pi(s^*) = \pi(s)^*$ and $\pi(0) = 0$.

$C_u^*(S)$ — universal for representations of S.

$C_u^*(S) = C^*(\mathcal{G}_u(S))$ for an étale groupoid $\mathcal{G}_u(S)$ constructed from S.

$\mathcal{G}_u(S)^{(0)}$ is homeomorphic to the space of filters in $E(S)$, and $C_0(\mathcal{G}_u(S)^{(0)}) = C^*(E(S))$ is always a commutative subalgebra of $C_u^*(S)$.

Example: 2×2 matrices

$\mathbb{M}_2(\mathbb{C})$ – 2×2 matrices over $\mathbb{C}$

e_{ij} = matrix with 1 in (i,j) entry, 0 elsewhere.

$$E(\mathcal{I}_2) = \{1_2, e_{11}, e_{22}, 0_2\}$$

Set of filters $= \{\{1_2\}, \{1_2, e_{11}\}, \{1_2, e_{22}\}\}$

$C(\mathcal{G}_u(S_2)^{(0)}) = \mathbb{C}^3$

Even though it "feels like" $C^*_u(\mathcal{I}_2)$ should be $\mathbb{M}_2(\mathbb{C})$, it cannot be.

$C^*_u(\mathcal{I}_2) \cong \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{C}$, with universal representation given by

$$\pi_u\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \pi_u\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{bmatrix} \textit{ else}$$

Example: 2×2 matrices

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$\pi_u(E(\mathcal{I}_2))$ is a commuting set of projections, and two commuting projections in a C*-algebra always have a join:

$$e \vee f = e + f - ef$$

$$\pi_u(e_{11}) \vee \pi_u(e_{22}) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \neq \pi_u(1_2) = \pi_u(e_{11} \vee e_{22})$$

If we want to recover $\mathbb{M}_2(\mathbb{C})$, we would like to look at representations which preserve joins.

Exel's tight representations

The above example was special, $E(\mathcal{I}_2)$ is a Boolean algebra, and has joins.

In general, $E(S)$ won't have joins.

$C \subset_{\text{fin}} E(S)$ is a cover for $e \in E(S)$ if for all $0 \neq f \leqslant e$, there is a $c \in C$ such that $fc \neq 0$.

Exel (08) introduced the notion of a tight representation.

π is tight if whenever C is a cover for e, we have $\bigvee_{c \in C} \pi(c) = \pi(e)$ $(\pm\epsilon)$

$C^*_{\text{tight}}(S)$ universal for tight representations.

$C^*_{\text{tight}}(\mathcal{I}_2) \cong \mathbb{M}_2(\mathbb{C})$

Example: Cuntz algebras

$\mathcal{O}_2$ is the universal C*-algebra generated by two isometries s_0, s_1 satisfying

$$s_0^* s_0 = 1 = s_1^* s_1$$

$$s_0^* s_1 = 0 = s_1^* s_0$$

$$s_0 s_0^* + s_1 s_1^* = 1$$

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It's generated by partial isometries, and one can arrange for a generating set of partial isometries closed under product and involution.

Example: Cuntz algebras

$X = \{0,1\}$ two element set, $X^0 = \{\varnothing\}$, X^n words of length n in X.

$$X^* = \bigcup_{n\geq 0} X^n$$

For $\alpha \in X^*$, let $s_\alpha := s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{|\alpha|}}$, and note $s_\alpha^* = s_{\alpha_{|\alpha|}}^* \cdots s_{\alpha_2}^* s_{\alpha_1}^*$

Let $s_\varnothing = 1$

$s_\alpha s_\beta = s_{\alpha\beta}$ and $s_\alpha^* s_\beta^* = s_{\beta\alpha}^*$

$P_2 = \{s_\alpha s_\beta^* \mid \alpha, \beta \in \{0,1\}^*\} \cup \{0\}$ polycyclic monoid

$$(s_\alpha s_\beta^*)(s_\gamma s_\nu^*) = \begin{cases} s_{\alpha\gamma'} s_\nu^* & \text{if } \gamma = \beta\gamma' \\ s_\alpha s_{\nu\beta'}^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise} \end{cases}$$

$E(P_2) = \{s_\alpha s_\alpha^* \mid \alpha \in \{0,1\}^*\} \cup \{0\}$

Example: Cuntz algebras

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$C_u^*(P_2) \cong \mathcal{T}_2$. This is the universal C*-algebra generated by elements as above, with the last relation removed.

$C^*_{\text{tight}}(P_2) \cong \mathcal{O}_2$

This is a recurring theme in the literature — $C^*_{\text{tight}}(S)$ defined by the combinatorics, with a corresponding "Toeplitz extension" $C_u^*(S)$.

Summary (so far)

Given a C*-algebra generated by partial isometries, one can often find a generating inverse semigroup of partial isometries S.

There is then a canonical universal C*-algebra $C_u^*(S)$, and a canonical boundary quotient $C^*_{\text{tight}}(S)$.

Both can be realized as groupoid C*-algebras, for the universal groupoid $\mathcal{G}_u$ and the tight groupoid $\mathcal{G}_{\text{tight}}(S)$ respectively.

Steinberg (2016) and Exel-Pardo (2016) found algebraic conditions on S which guarantee certain properties of $\mathcal{G}_{\text{tight}}(S)$

Simplicity of C*-algebras of inverse semigroups

Theorem (Renault, Brown-Clark-Farthing-Sims)

Let $\mathcal{G}$ be a Hausdorff étale groupoid. Then $C^*(\mathcal{G})$ is simple if and only if

1. *$\mathcal{G}$ is minimal (every orbit is dense)*
2. *$\mathcal{G}$ is effective (the interior of the isotropy group bundle is the unit space), and*
3. *$\mathcal{G}$ satisfies weak containment ($C_r^*(\mathcal{G}) \cong C^*(\mathcal{G})$).*

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For example, the following guarantees Hausdorff:

For $s \in S$, let $\mathcal{J}_s = \{e \in E(S) \mid e \leq s\}$

$\mathcal{G}_{\text{tight}}(S)$ Hausdorff iff $\mathcal{J}_s$ has a finite cover for all s

Stronger: S is E*-unitary if $s \notin E(S) \Rightarrow \mathcal{J}_s = \{0\}$

C*-algebras of inverse semigroups

Many C*-algebras have been identified as the tight C*-algebra of a generating inverse semigroup:

- (k)-Graph C*-algebras (Exel 2008)
- Tiling C*-algebras (Kellendonk 97, Lenz 11, Exel-Gonçalves-S 2012)
- Self-similar group C*-algebras (Exel-Pardo 2016)
- Katsura algebras (Exel-Pardo 2016)
- C*-algebras of right LCM semigroups (S 2015)
- Carlsen-Matsumoto subshift algebras (S 2015) ($*$)
- AF C*-algebras (Renault 1980, Lawson-Scott 2014, S 2016)
- Any C*-algebra of an ample étale groupoid (Exel 2010)
- C*-algebras of Boolean dynamical systems (Carlsen-Ortega-Pardo 2016)
- C*-algebras of labeled spaces (Boava-de Castro-Mortari 2016)

Some questions that we can answer

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Question 2: Given a C*-algebra universal for some set of partial isometries, what is a suitable boundary quotient?

Answer: $C^*_{\text{tight}}(S)$ for some S, of course!

Example: Li's Semigroup C*-algebras

P countable semigroup

Left cancellative: $ps = pq \Rightarrow s = q$

Note that left cancellative inverse semigroups are groups, and if we assume that ISGs have a zero, then no ISGs are left cancellative.

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Principal right ideal: $rP = \{rq \mid q \in P\}$

Elements of rP are right multiples of r

Assume $1 \in P$ (ie, P is a monoid)

Example: Li's Semigroup C*-algebras

Study P by representing on a Hilbert space, similar to groups.

$\ell^2(P)$ – square-summable complex functions on P.

δ_x – point mass at $x \in P$. Orthonormal basis of $\ell^2(P)$.

$v_p : \ell^2(P) \to \ell^2(P)$ bounded operator $v_p(\delta_x) = \delta_{px}$ (necessarily isometries)

$\{v_p\}_{p \in P}$ generate the reduced C*-algebra of P, $C_r^*(P)$

$v : P \to C_r^*(P)$ is called the left regular representation

Unlike the group case, considering all representations turns out to be a disaster

Li (Nica): we have to care for ideals.

Example: Li's Semigroup C*-algebras

For $X \subset P$, then $e_X : \ell^2(P) \to \ell^2(P)$ is defined by

$$(e_X\xi)(p) = \begin{cases} \xi(p) & \text{if } p \in X \\ 0 & \text{otherwise.} \end{cases}$$

Note: $v_1 = e_P$

Note that in $\mathcal{B}(\ell^2(P))$,

$$v_p e_X v_p^* = e_{pX} \qquad v_p^* e_X v_p = e_{p^{-1}X}$$

If $p \in P$ and X is a right ideal, then

$$pX = \{px \mid x \in X\} \quad p^{-1}X = \{y \mid py \in X\}$$

are right ideals too.

Example: Li's Semigroup C*-algebras

$$pX = \{px \mid x \in X\} \qquad p^{-1}X = \{y \mid py \in X\}$$

$\mathcal{J}(P)$ – smallest set of right ideals containing P, $\emptyset$, and closed under finite intersection and the above operations – constructible ideals.

These are the ideals which are "constructible" inside $C_r^*(P)$.

1. $e_X e_Y = e_{X \cap Y}$
2. $e_P = 1$, $e_\emptyset = 0$
3. $v_p e_X v_p^* = e_{pX}$ and $v_p^* e_X v_p = e_{p^{-1}X}$

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Definition (Li)

$C^*(P)$ is the universal C*-algebra generated by isometries $\{v_p \mid p \in P\}$ and projections $\{e_X \mid X \in \mathcal{J}(P)\}$ satisfying the above (and $v_p v_q = v_{pq}$).

Example: Li's Semigroup C*-algebras

Norling (2014): for each $p \in P$, let

$$\lambda_p : P \to P$$

$$q \mapsto pq$$

Each is a bijection between subsets of P, i.e. $\lambda_p \in \mathcal{I}(P)$

The inverse semigroup they generate $\mathcal{I}_l(P)$, the left inverse hull of P, generates $C^*(P)$.

$$C^*(P) \cong C^*_u(\mathcal{I}_l(P))$$

Boundary Quotient

Simplification: suppose that for all $r, q \in P$, either $rP \cap qP = \emptyset$ or

$$rP \cap qP = sP \text{ some } s \in P$$

Then $\mathcal{J}(P) = \{sP \mid s \in P\} \cup \{\emptyset\}$.

Such semigroups are called Clifford semigroups, or right LCM semigroups.

Finite $F \subset P$ is a foundation set if for all $r \in P$, there is $f \in F$ with $fP \cap rP \neq \emptyset$.

Definition (Brownlowe, Ramagge, Robertson, Whittaker)

The boundary quotient $\mathcal{Q}(P)$ is the universal C*-algebra generated by the same elements and relations as in Li's $C^*(P)$, and also satisfying

$$\prod_{f \in F}(1 - e_{fP}) = 0 \text{ for all foundation sets } F.$$

Boundary Quotient

For a right LCM monoid P,

$$\mathcal{I}_l(P) := \{\lambda_p\lambda_q^* \mid p, q \in P\} \cup \{0\}$$

$$(\lambda_p\lambda_q^*)(\lambda_r\lambda_s^*) = \begin{cases} \lambda_{pq'}\lambda_{sr'}^* & \text{if } qP \cap rP = kP \text{ and } qq' = rr' = k \\ 0 & \text{if } qP \cap rP = \emptyset \end{cases}$$

Theorem (S, 2015)

$\mathcal{Q}(P) \cong C^*_{tight}(\mathcal{I}_l(P))$

Used results of Steinberg and Exel-Pardo to find conditions on P which guarantee that $\mathcal{Q}(P)$ is simple and purely infinite.

Example: Free Semigroups

Let X be the two element set, and again let X^* be the set of finite words.

This is the free semigroup on X, under concatenation.

Right LCM:

$$\alpha X^* \cap \beta X^* = \begin{cases} \alpha X^* & \alpha = \beta\gamma \\ \beta X^* & \beta = \alpha\gamma \\ \emptyset & \text{otherwise} \end{cases}$$

$C^*(X^*) \cong C^*_r(X^*) \cong \mathcal{T}_2$

$\mathcal{I}_l(X^*) = P_2$

$\mathcal{Q}(X^*) \cong C^*_{\text{tight}}(P_2) \cong \mathcal{O}_2$

LCM Semigroups

What I describe now is work in progress with Ilija Tolich.

In Li's C*-algebra, the left multiplicative structure is emphasized.

Many left cancellative semigroups are also right cancellative.

Is there a nice construction of a C*-algebra from such a semigroup that incorporates both the left and the right structure?

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Tolich (2017): doubly quasi-lattice ordered groups

LCM Semigroups

Definition

P is an LCM semigroup if

- P is cancellative (left and right)
- $pP \cap qP = rP$ for some $r \in P$, or is empty
- $Pp \cap Pq = Pk$ for some $k \in P$, or is empty

Example: Free semigroups

Example: Zappa-Szép products associated to self-similar groups

LCM Semigroups

Fix $a \in P$, and let $I_a = \{x \in P \mid Pa \subset Px\}$

Define $v^a : P \to B(\ell^2(I_a))$ by

$$v_q^a(\delta_x) = \begin{cases} \delta_{qx} & qx \in I_a \\ 0 & \text{otherwise} \end{cases}$$

Definition

The reduced C*-algebra of the LCM semigroup P, $C_r^*(P, P^{\mathrm{op}})$, is the C*-algebra generated by the image of $v = \bigoplus_{a \in P} v^a$

We also define an universal C*-algebra analogous to Li's, caring for "constructible subsets" of $P \times P$.

LCM Semigroups

$\{v_q v_q^* \mid q \in P\} \cong \{qP \mid q \in P\}$

$\{v_q^* v_q \mid q \in P\} \cong \{Pq \mid q \in P\}$ (as semilattices)

$(v_r v_r^*)(v_q v_q^*) = v_k v_k^*$ where $rP \cap qP = kP$

$(v_r^* v_r)(v_q^* v_q) = v_m^* v_m$ where $Pp \cap Pq = Pm$

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Can show: $S = \{v_p v_q^* v_r \mid p, q, r \in P\} \cup \{0\}$ is an inverse semigroup of partial isometries.

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Can show: $S = \{v_p v_q^* v_r \mid p, q, r \in P\} \cup \{0\}$ is an inverse semigroup of partial isometries.

Boundary quotient — the tight C*-algebra of this inverse semigroup.

Free semigroups: $C^*_{\text{tight}}(S) \cong C(X^{\mathbb{Z}}) \rtimes_\sigma \mathbb{Z}$, where σ is the left shift.

So while free semigroups gave us Cuntz algebras which are simple and purely infinite, here they give us something not simple and stably finite.

Inverse Semigroup – C*-algebra dictionary

Inverse semigroup	**C*-algebra**	**Groupoid**
$s \in S$	partial isometry	compact open bisection
$e \in E(S)$	projection	compact open set of units
Green's relation $\mathcal{D}$	Murray-von Neumann equivalence	
Type monoid	K-theory	Type Monoid
Boolean inverse monoid		**all** compact open bisections
Invariant mean	trace	invariant measure

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