# C*-algebras generated by partial product systems over $\mathbb{N}$

Devarshi Mukherjee (joint work with Ralf Meyer)

Mathematisches Institut, Georg-August-Universität Göttingen

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# Outline

# Introduction

1. Several important classes of C*-algebras arise as Cuntz-Pimnser algebras of correspondences.
2. General idea: Given a C*-algebra $A$, C*-correspondences $\mathcal{E}_n$, define suitable *representations* of $(A, \mathcal{E}_n)$ in a C*-algebra.
3. Compute the *universal C*-algebra* $\mathcal{T}_{\mathcal{E}_n}$ for such representations.
4. Divide out a 'nice' ideal $\mathcal{I}$ and obtain the *Cuntz–Pimsner algebra* $\mathcal{O}_{\mathcal{E}_n} = \mathcal{T}_{\mathcal{E}_n}/\mathcal{I}$.

# Weak Partial Product Systems

- Definition

  A weak partial product system consists of a C*-algebra $A$, $A, A$-correspondences $\mathcal{E}_n$ for all $n \in \mathbb{N}$, and isometries $\mu_{n,m}\colon \mathcal{E}_n \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_{n+m}$ for all $n, m \in \mathbb{N}$, subject to several conditions.

- Proposition

  *A weak partial product system $(A, \mathcal{E}_n, \mu_{n,m})$ is equivalent to a bicategorical morphism from the bicategory $\mathbb{N}$ to the correspondence bicategory $Corr_{\subseteq}$ with isometries as 2-arrows.*

# Weak representations

## Definition

A *weak representation* of a weak partial product system $(A, \mathcal{E}_n, \mu_{n,m})$ in a C*-algebra $B$ consists of linear maps $\omega_n \colon \mathcal{E}_n \to B$ and a nondegenerate $\star$-homomorphism $\omega_0 \colon A \to B$ subject to the following data:

1. $\omega_n(a \cdot x) = \omega_0(a)\omega_n(x)$ for all $x \in \mathcal{E}_n$, $n \in \mathbb{N}$.
2. $\omega_n(x)\omega_m(y) = \omega_{n+m}(\mu_{n,m}(x \otimes y))$ for $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$.
3. $\omega_n(x)^\star\omega_n(y) = \omega_0(\langle x, y \rangle)$, for all $x, y \in \mathcal{E}_n$, $n \in \mathbb{N}$.

# Technicalities

- Product systems over $\mathbb{N}$ satisfy *compact alignment*, which yields a 'reasonable' $\star$-algebra generated by images of representations.
- No compact alignment or similar nice conditions in the case of partial product systems.
- We impose an additional condition that gives a similar outcome as compact alignment.

# Representations

## Definition

A weak representation is called a *representation* if it additionally satisfies the following conditions:

1. $\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) \subseteq \omega_{m-n}(\mathcal{E}_{m-n}) \cdot B$ for all $n, m \in \mathbb{N}$ with $m > n$;
2. $\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) \subseteq \omega_{n-m}(\mathcal{E}_{n-m})^* \cdot B$ for all $n, m \in \mathbb{N}$ with $m < n$.

# Fock representation

- For $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, define $S_{n,m}(x)\colon \mathcal{E}_m \to \mathcal{E}_{n+m}$, $y \mapsto \mu_{n,m}(x \otimes y)$.
- These maps combine to give linear operators $S_n\colon \mathcal{E}_n \to \mathcal{L}(\bigoplus_{m\in\mathbb{N}} \mathcal{E}_m)$.
- In examples arising from graph C*-algebras, the maps $S_n(x)$ are even adjointable (so that they are a weak representation in $\mathcal{B}(\bigoplus_{n\in\mathbb{N}} \mathcal{E}_n)$).

# Partial product system

## Definition

A *partial product system* is a weak partial product system with the extra property that the Fock representation exists and is a representation.

# Reformulation

## Lemma

*The additional conditions for a representation are equivalent to the following equations:*

$$\omega_n(x)^*\omega_m(y) = \begin{cases} \omega_{m-n}(S_n(x)^*y) & \text{if } m > n, \\ \omega_{n-m}(S_m(y)^*x)^* & \text{if } n > m. \end{cases}$$

*where* $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$.

# Example

- $(V, E_n, r_n, s_n)$ a sequence of graphs, $V$ a countable, discrete set and $E_0 = V$ and $r_0 = s_0 = \mathrm{id}_V$.
- $A = \mathrm{C}_0(V)$,
- $\mathcal{E}_n$ is the completion of $\mathrm{C_c}(E_n)$ in a certain inner product. As a vector space, it is generated by the characteristic functions $(\delta_x)_{x \in E_n}$.
- The composition $\mathcal{E}_n \otimes_{\mathrm{C}_0(V)} \mathcal{E}_m$ is the correspondence associated to the graph $(V, E_n \times_{s,r} E_m, s(f,g) := s_m(g), r(f,g) := r_n(f))$.

# Example continued

- Injective maps $\tilde{\mu}_{n,m}\colon E_n \times_{s,r} E_m \to E_{n+m}$, written multiplicatively as $x \cdot y := \tilde{\mu}_{n,m}(x, y)$ for $x \in E_n$, $y \in E_m$ with $s_n(x) = r_m(y)$
- These induce isometries $\mu_{n,m}(\delta_x \otimes \delta_y) = \delta_{x \cdot y}$
- Need $s(x \cdot y) = s(y)$ and $r(x \cdot y) = r(x)$ for technical reasons (associativity and bimodularity).

# Example continued

- Define

$$S_n : \mathcal{E}_n \to \mathbb{B}(\mathcal{F}), \text{ where } \mathcal{F} \text{ is the Fock module ;}$$
$$S_n(\delta_x)(\delta_y) = \mu_{n,m}(\delta_x \otimes \delta_y) = \delta_{x \cdot y}, \text{ for } x \in E_n, y \in E_m.$$

- Its adjoint is given by

$$T_n(\delta_x)(\delta_y) = \begin{cases} \langle \delta_x \mid \delta_{y_n} \rangle \cdot \delta_{y_m} & \text{if } y = y_n \cdot y_m, \\ 0 & \text{else} \end{cases}$$

for $x \in E_n, y \in E_{n+m}, y_n \in E_n, y_m \in E_m$.

# Toeplitz algebra

## Definition

Given a partial product system $(A, (\mathcal{E}_n)_n, \mu_{n,m})$, its Toeplitz algebra is a C$^*$-algebra $\mathcal{T}$ together with a representation $(\bar{\omega}_n, \bar{\omega}_0)_n$ that is universal in the sense that given any other representation $(\omega_n, \omega_n)_n$ of $(A, (\mathcal{E}_n)_n, \mu_{n,m})$ in a C$^*$-algebra $D$, there exists a unique nondegenerate $^*$-homomorphism $\varrho\colon \mathcal{T} \to D$ such that $\varrho \circ \bar{\omega}_n = \omega_n$ for each $n \in \mathbb{N}$.

# Existence

## Proposition

*Any partial product system $(A, (\mathcal{E}_n)_n, \mu_{n,m})$ has a Toeplitz algebra. It is the universal* C*-*algebra generated by the symbols $\bar{\omega}_n(x)$ for $x \in \mathcal{E}_n$, $n \in \mathbb{N}$, subject to the following relations:*

1. *the maps $x \mapsto \bar{\omega}_n(x)$ are linear for each $n \in \mathbb{N}$;*
2. $\bar{\omega}_0$ *is a* $^*$-*homomorphism;*
3. $\bar{\omega}_n(x)\bar{\omega}_m(y) = \bar{\omega}_{n+m}(\mu_{n,m}(x \otimes y))$ *for* $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$, $n, m \in \mathbb{N}$;
4. $\bar{\omega}_n(x)^*\bar{\omega}_n(y) = \bar{\omega}_0(\langle x \,|\, y\rangle_A)$ *for all* $x \in \mathcal{E}_n$, $y \in \mathcal{E}_n$ $n \in \mathbb{N}$;
5. $\omega_n(x)^*\omega_m(y) = \omega_{m-n}(S_n(x)^*(y))$, *for* $m > n$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$.

# Results

## Lemma

$$\mathcal{T} = \overline{\text{span}}\{\bar{\omega}_n(x)\bar{\omega}_m(y)^* : x \in \mathcal{E}_n, y \in \mathcal{E}_m, n, m \in \mathbb{N}\},$$

*where $(\bar{\omega}_n, \bar{\omega}_0)$ is the universal Toeplitz representation.*

## Proposition

*Using the gauge action and its 'spectral projections', we can describe the Toeplitz algebra as a $\mathbb{Z}$–graded C*-algebra $\mathcal{T} = \overline{\bigoplus_{n\in\mathbb{Z}} \mathcal{T}_n}$*

## Definition

There is a unique *-homomorphism $\Theta_{n,m} \colon \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m) \to B$ with

$$\Theta_{n,m}(|x\rangle\langle y|) = \omega_n(x)\omega_m(y)^* \qquad (4)$$

for all $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$

# Results

**Theorem**

*The zero-fiber $\mathcal{T}_0$ is the inductive limit of the inductive system $(\sum_{j=0}^{N} \bar{\Theta}_j(\mathbb{K}(\mathcal{E}_j)))_{N\in\mathbb{N}}$.*

**Theorem**

*The Fock representation $(S_n)_{n\in\mathbb{N}}$ on the Fock module $\mathcal{F}$ over $A$ induces a faithful representation of $\mathcal{T}$. So $\mathcal{T}$ is isomorphic to the $\mathrm{C}^*$-subalgebra of $\mathbb{B}(\mathcal{F})$ generated by $S_n(\mathcal{E}_n)$ for all $n \in \mathbb{N}$.*

# Cuntz–Pimsner algebra



## Definition

(Cuntz–Pimsner covariance) Let $(I_n)_n \in \mathbb{N}_{\geq 1}$ be ideals in $A$ with $\varphi_n(I_n) \subseteq \mathbb{K}(\mathcal{E}_n)$. A representation $(\omega_n, \omega_0)$ is said to be *Cuntz–Pimsner covariant* on $(I_n)_n \in \mathbb{N}_{\geq 1}$ if $\omega_0(a) = \Theta_n(\varphi_n(a))$ for all $a \in I_n$ and each $n \in \mathbb{N}_{\geq 1}$.

## Proposition

*A Toeplitz representation $(\omega_n, \omega_0)_{n\in\mathbb{N}}$ of a partial product system $(A, \mathcal{E}_n)_n$ in a C*-algebra $B$ is Cuntz–Pimsner covariant on ideals $I_n$ if and only if $\omega_0(I_n) \subseteq \omega_n(\mathcal{E}_n)B$ for each $n \in \mathbb{N}$.*

## Proposition

*Given a partial product system $(\mathcal{E}_n)_{n\in\mathbb{N}}$, there exists a* $\mathrm{C}^*$*-algebra* $\mathcal{O}(\mathcal{E}_n)$ *with a representation* $(\tilde{\omega}_n, \tilde{\omega}_0)_n$ *that is universal for Cuntz–Pimsner covariant Toeplitz representations in the following sense: if* $(\tilde{\alpha}_n, \tilde{\alpha}_0)_n$ *is another representation of the given partial product system on a* $\mathrm{C}^*$*-algebra* $B$*, then there exists a unique* $^*$*-homomorphism* $\tilde{\varrho}\colon \mathcal{O}(\mathcal{E}_n) \to B$ *such that* $\tilde{\varrho}(\tilde{\omega}_n(x)) = \tilde{\alpha}_n(x)$ *for each* $n \in \mathbb{N}$.

# Other Cuntz–Pimsner relations

### Lemma

*For $n, m \in \mathbb{N}$, let $J_m = \varphi_m^{-1}(\mathbb{K}(\mathcal{E}_m))$ and $T \in \mathbb{K}(\mathcal{E}_n)$. Then $T \otimes 1_m \in \mathbb{K}(\mathcal{E}_n \otimes_A \mathcal{E}_m)$ if and only if $T \in \mathbb{K}(\mathcal{E}_n J_m)$.*

Let $\varphi_{n,m}\colon T \mapsto T \otimes 1_m$ for $T \in I_{n,m} \subseteq \mathbb{K}(\mathcal{E}_n J_m)$.

### Definition

A Toeplitz representation $(\omega_n, \omega_0)$ is said to be *strongly Cuntz–Pimsner covariant* if $\Theta_n(T) = \Theta_{n+m}(\varphi_{n,m}(T))$ for all $T \in I_{n,m} \trianglelefteq \mathbb{K}(\mathcal{E}_n)$ with $I_{n,m} \subseteq \mathbb{K}(\mathcal{E}_n J_m)$ and for each $n, m \in \mathbb{N}$.

### Lemma

*A Toeplitz representation $(\omega_n, \omega_0)_{n\in\mathbb{N}}$ of a partial product system $(A, \mathcal{E}_n)_n$ in a C*-algebra $B$ is strongly Cuntz–Pimsner covariant on ideals $I_{n,m}$ of $\mathbb{K}(\mathcal{E}_n)$ contained in $\mathbb{K}(\mathcal{E}_n J_m)$, if and only if $\Theta_n(I_{n,m}) \subseteq \omega_{n+m}(\mathcal{E}_{n+m})B$ for each $n, m \in \mathbb{N}$.*

- By the Rieffel correspondence, ideals $I_{n,m}$ correspond uniquely to ideals $J_{n,m}$ in $A$

## Theorem

*Strong Cuntz–Pimsner covariance on an ideal $I_{n,m}$ in $\mathbb{K}(\mathcal{E}_n)$ acting by compact operators on $\mathcal{E}_{n+m}$ is equivalent to Cuntz–Pimsner covariance on an ideal $J_{n,m}$ in $A$, acting by compact operators on $\mathcal{E}_m$.*

# Fell bundles and their representations

## Definition

A *Fell bundle* over a group $G$ is a collection $\mathcal{B} = (B_g)_{g \in G}$ of Banach spaces together with a multiplication and an involution map on the total space $B = \sqcup_{g \in G} B_g$,

$$\cdot\colon B \times B \to B, \quad {}^*\colon B \to B,$$

satisfying the following:

1. $B_g B_h \subseteq B_{gh}$,
2. the multiplication $B_g \times B_h \to B_{gh}$ is bilinear,
3. multiplication on $B$ is associative,
4. $\|bc\| \leq \|b\| \, \|c\|$,
5. $(B_g)^* \subseteq B_{g^{-1}}$,
6. involution is conjugate linear from $B_g$ to $B_{g^{-1}}$,
7. $(bc)^* = c^* b^*$,
8. $(b^*)^* = b$,

# Fell bundles and their representations

## Definition

A $^*$-*representation* of a Fell bundle $\mathcal{B} = (B_g)_{g \in G}$ in a C$^*$-algebra $D$ consists of linear maps $\omega_g \colon B_g \to D$ such that

1. $\omega_1 \colon B_1 \to D$ is a nondegenerate $^*$-homomorphism,
2. $\omega_g(b)\omega_h(c) = \omega_{gh}(bc)$,
3. $\omega_{g^{-1}}(b^*) = \omega_g(b)^*$

for all $g, h \in G$, $b \in B_g$, $c \in B_h$.

## Proposition

*Any Fell bundle $\mathcal{B} = (B_g)_{g\in G}$ has a section* C*-*algebra denoted by $C^*(\mathcal{B})$. It is the universal* C*-*algebra generated by the symbols $\omega_g(b)$ for $b \in B_n, g \in G$, subject to the following relations:*

1. *the maps $b \mapsto \omega_g(b)$ are linear for each $g \in G$;*
2. *$\omega_1$ is a $^*$-homomorphism;*
3. *$\omega_g(b)\omega_h(c) = \omega_{gh}(bc)$ for $b \in B_g, c \in B_h, g, h \in G$;*
4. *$\omega_{g^{-1}}(b^*) = \omega_g(b)^*$,*

*for all $b \in B_g, g \in G$.*

# Relation between Cuntz–Pimsner and section C*-algebra

## Proposition

*Any Cuntz–Pimsner covariant Toeplitz representation* $(\omega_n, \omega_0)_{n\in\mathbb{N}}$ *in a* $\mathrm{C}^*$*-algebra* $D$ *extends to a Fell bundle representation* $(\omega_n, \omega_0)_{n\in\mathbb{Z}}$ *in* $D$*.*

## Proposition

*The restriction of a Fell bundle over* $\mathbb{Z}$ *to* $\mathbb{N}$ *is a partial product system consisting of Hilbert bimodules. Conversely, any partial product system of Hilbert bimodules extends to a Fell bundle over* $\mathbb{Z}$*.*

## Theorem

*The section* $\mathrm{C}^*$*-algebra of the Fell bundle over* $\mathbb{Z}$ *is canonically and* $\mathbb{T}$*-equivariantly isomorphic to the Cuntz–Pimsner algebra of the corresponding partial product system with respect to the sequence of Katsura's ideals.*

Thank you.