On Exel-Pardo algebras as C^*-algebras associated with left cancellative small categories

Oslo, December 8, 2017

Joint work with

S. Kaliszewski, John Quigg and Jack Spielberg

Exel-Pardo systems I

Throughout, G will denote a discrete group.

Let $E = (E^0, E^1, r, s)$ be a directed graph and assume

$G \curvearrowright E$, that is, $G \curvearrowright E^0$, $G \curvearrowright E^1$, and we have

$$r(g \cdot e) = g \cdot r(e) \quad s(g \cdot e) = g \cdot s(e) \quad \text{for all } g \in G, e \in E^1.$$

Definition

A *graph cocycle* for $G \curvearrowright E$ is a function $\varphi : G \times E^1 \to G$ satisfying

- $\varphi(gh, e) = \varphi(g, h \cdot e)\, \varphi(h, e) \quad$ for all $g, h \in G, e \in E^1$,
- $\varphi(g, e) \cdot s(e) = g \cdot s(e) \quad$ for all $g \in G, e \in E^1$,

The triple (E, G, φ) is then called an *Exel-Pardo system*.

To each Exel-Pardo system (E, G, φ) one may attach

- a Toeplitz algebra $\mathcal{T}(E, G, \varphi)$;
- a Cuntz-Pimsner algebra $\mathcal{O}(E, G, \varphi)$, that we call the *Exel-Pardo algebra* of (E, G, φ).

Cf. Exel and Pardo (2017) when E is finite and sourceless. Otherwise, see BKQ (2017).

- The case where $E^0 = \{v_1\}$ (and E^1 is finite) includes Nekrashevych's C^*-algebras associated with self-similar (actions of) groups.
- Any Katsura algebra $\mathcal{O}_{A,B}$ is an Exel-Pardo algebra of the form $\mathcal{O}(E_A, \mathbb{Z}, \varphi_{A,B})$.
- Laca, Raeburn, Ramagge and Whittaker have recently introduced self-similar groupoid actions on finite directed graphs.

Assume that E is row-finite (i.e., $|r^{-1}(v)| < \infty$ for all $v \in E^0$). Then $\mathcal{T}(E, G, \varphi)$ and $\mathcal{O}(E, G, \varphi)$ may described via generators and relations.

Definition

A (nondegenerate) *representation* of (E, G, φ) in a C^*-algebra B is a triple (P, S, U), where

- $\{P_v,\ S_e : v \in E^0, e \in E^1\}$ is a Toeplitz E-family in B,
- $\sum_{v \in E^0} P_v$ converges strictly to $I_{M(B)}$,
- $U\colon G \to M(B)$ is a unitary homomorphism,
- for all $g \in G$, $v \in E^0$, and $e \in E^1$ we have

$$U_g P_v = P_{g \cdot v} U_g\ , \quad U_g S_e = S_{g \cdot e} U_{\varphi(g,e)}.$$

Such a representation (P, S, U) is called *covariant* if $\{P_v,\ S_e : v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family in B.

Theorem

There exists a representation (p, s, u) *of* (E, G, α) *in* $\mathcal{T}(E, G, \alpha)$ *such that*

- $\mathcal{T}(E, G, \alpha)$ *is generated as a* C^**-algebra by*

$$\{p_v u_g : v \in E^0, g \in G\} \cup \{s_e u_g : e \in E^1, g \in G\};$$

- *for every representation* (P, S, U) *of* (E, G, φ) *in a* C^**-algebra* B*, there exists a nondegenerate homomorphism* $\phi = \phi_{P,S,U} : \mathcal{T}(E, G, \alpha) \to B$ *such that*

$$P = \phi \circ p, \quad S = \phi \circ s, \quad U = \overline{\phi} \circ u\,.$$

Further, there also exists a covariant representation $(\tilde{p}, \tilde{s}, \tilde{u})$ *of* (E, G, α) *in* $\mathcal{O}(E, G, \alpha)$ *with similar properties.*

Finitely aligned left cancellative small categories

Let $\mathcal{C}$ be a left cancellative small category. We consider the set of objects $\mathcal{C}^0$ as a subset of $\mathcal{C}$, and let $r, s : \mathcal{C} \to \mathcal{C}^0$ denote the range and source maps. For $\alpha \in \mathcal{C}$, we set

$$\alpha\mathcal{C} = \{\alpha\beta : \beta \in \mathcal{C},\, s(\alpha) = r(\beta)\}$$

For $\alpha, \alpha' \in \mathcal{C}$, write $\alpha \sim \alpha'$ whenever there is an invertible $\delta \in \mathcal{C}$ such that $r(\delta) = s(\alpha)$ and

$$\alpha' = \alpha\delta.$$

Note that this gives an equivalence relation on $\mathcal{C}$, and that

$$\alpha \sim \alpha' \quad \text{if and only if} \quad \alpha\mathcal{C} = \alpha'\mathcal{C}\,.$$

Definition

We say $\mathcal{C}$ is *singly aligned* if for all $\alpha, \beta \in \mathcal{C}$,

either $\alpha\mathcal{C} \cap \beta\mathcal{C} = \varnothing$ or there is $\gamma \in \mathcal{C}$ such that

$$\alpha\mathcal{C} \cap \beta\mathcal{C} = \gamma\mathcal{C}\,.$$

Note that γ is then unique, up to equivalence.

More generally, we say $\mathcal{C}$ is *finitely aligned* if for all $\alpha, \beta \in \mathcal{C}$,

either $\alpha\mathcal{C} \cap \beta\mathcal{C} = \varnothing$ or there is a finite nonempty subset $L \subseteq \mathcal{C}$ such that

$$\alpha\mathcal{C} \cap \beta\mathcal{C} = \bigcup_{\gamma \in L} \gamma\mathcal{C}\,.$$

Note that in the equation

$$\alpha\mathcal{C} \cap \beta\mathcal{C} = \bigcup_{\gamma\in L} \gamma\mathcal{C}\,, \tag{1}$$

the finite nonempty set L can be chosen to be *minimal* (or *independent*), in the sense that

$$\gamma_1, \gamma_2 \in L, \gamma_1 \neq \gamma_2 \quad \Rightarrow \quad \gamma_2 \notin \gamma_1\mathcal{C}.$$

Moreover, if we say that $L, L' \subseteq \mathcal{C}$ are equivalent whenever we have

$$L' \subseteq \bigcup_{\gamma\in L} [\gamma] \quad \text{and} \quad L \subseteq \bigcup_{\gamma'\in L'} [\gamma']\,,$$

then the set L is unique, up to equivalence, among all finite minimal subsets of $\mathcal{C}$ satisfying (1).

Assume $\mathcal{C}$ is finitely aligned and let $\alpha, \beta \in \mathcal{C}$.

- If $\alpha\mathcal{C} \cap \beta\mathcal{C} = \varnothing$, we set $\alpha \vee \beta := \varnothing$.
- Otherwise, we set $\alpha \vee \beta := L$,
  where L is a finite, minimal subset of $\mathcal{C}$ such that
  $$\alpha\mathcal{C} \cap \beta\mathcal{C} = \bigcup_{\gamma \in L} \gamma\mathcal{C} \, ,$$
  having in mind that L is then only determined up to equivalence.

Definition

If $v \in \mathcal{C}^0$ and $F \subseteq v\mathcal{C}$, then F is said to be *exhaustive* at v if for every $\alpha \in v\mathcal{C}$ there is $\beta \in F$ with $\alpha\mathcal{C} \cap \beta\mathcal{C} \neq \varnothing$.

We let $\mathcal{C}$ denote a finitely aligned left cancellative small category.

Definition

A *representation* of $\mathcal{C}$ in a C^*-algebra B is a map $T\colon \mathcal{C} \to B$ such that for all $\alpha, \beta \in \mathcal{C}$, we have

1. $T_\alpha^* T_\alpha = T_{s(\alpha)}$;
2. $T_\alpha T_\beta = T_{\alpha\beta}$ if $s(\alpha) = r(\beta)$;
3. $T_\alpha T_\alpha^* T_\beta T_\beta^* = \bigvee_{\gamma \in \alpha \vee \beta} T_\gamma T_\gamma^*$.

Such a representation is called *covariant* if it also satisfies:

4. $T_v = \bigvee_{\alpha \in F} T_\alpha T_\alpha^*$ for every $v \in \mathcal{C}^0$ and every finite exhaustive set F at v.

Remarks

Let T is a representation of $\mathcal{C}$ in B.

- T_v is a projection in B for every $v \in \mathcal{C}^0$.
- If $v, w \in \mathcal{C}^0$ and $v \neq w$, then $T_v T_w = 0$.
- T_α is a partial isometry in B for every $\alpha \in \mathcal{C}$.
- We say that T is *nondegenerate* if the series $\sum_{v \in \mathcal{C}^0} T_v$ converges strictly to 1 in $M(B)$.
- We let $C^*(T)$ denote the C^*-subalgebra of B generated by the range of T. If we consider T as a representation of $\mathcal{C}$ in $C^*(T)$, then T is nondegenerate.
- It can be shown that if T is nondegenerate, then T is covariant if and only if T is *tight* as defined by Exel (in 2011 for representations of semigroupoids).

Theorem

Let B be a C^*-algebra, and let $T : \mathcal{C} \to B$. Then T is a representation of $\mathcal{C}$ if and only if it satisfies the following conditions:

- T_α *is a partial isometry for every* $\alpha \in \mathcal{C}$;
- *for all* $\alpha, \beta \in \mathcal{C}$, $T_\alpha T_\beta = \begin{cases} T_{\alpha\beta} & \text{if } s(\alpha) = r(\beta), \\ 0 & \text{otherwise}; \end{cases}$
- *the family of initial projections* $\{T_\alpha^* T_\alpha : \alpha \in \mathcal{C}\}$ *commutes, as does the family of final projections* $\{T_\alpha T_\alpha^* : \alpha \in \mathcal{C}\}$;
- $T_\alpha T_\alpha^* T_\beta T_\beta^* = 0$ *if* $\alpha\mathcal{C} \cap \beta\mathcal{C} = \varnothing$;
- $T_\alpha^* T_\alpha \geq T_\beta T_\beta^*$ *if* $s(\alpha) = r(\beta)$;
- $T_\alpha T_\alpha^* T_\beta T_\beta^* = \bigvee_{\gamma \in \alpha \vee \beta} T_\gamma T_\gamma^*$ *for all* $\alpha, \beta \in \mathcal{C}$.

Note: The first five conditions say that that T is a representation of $\mathcal{C}$ in the sense of Exel.

As follows from Spielberg's recent work, there exists a C^*-algebra $\mathcal{T}(\mathcal{C})$, called the *Toeplitz algebra* of $\mathcal{C}$, and a nondegenerate representation t of $\mathcal{C}$ in $\mathcal{T}(\mathcal{C})$ such that

- $\mathcal{T}(\mathcal{C}) = C^*(t)$;
- For every representation $T : \mathcal{C} \to B$, there is a unique homomorphism $\phi_T \colon \mathcal{T}(\mathcal{C}) \to B$ such that

$$T = \phi_T \circ t.$$

Moreover, it also follows that there exist a C^*-algebra $\mathcal{O}(\mathcal{C})$, called the *Cuntz-Krieger algebra* of $\mathcal{C}$, and a nondegenerate covariant representation $\tilde{t} : \mathcal{C} \to \mathcal{O}(\mathcal{C})$ such that

- $\mathcal{O}(\mathcal{C}) = C^*(\tilde{t})$,
- For every covariant representation $T : \mathcal{C} \to B$, there is a unique homomorphism $\psi_T \colon \mathcal{O}(\mathcal{C}) \to B$ such that

$$T = \psi_T \circ \tilde{t}.$$

Category systems and Zappa-Szép products

Let $\mathcal{C}$ be a small category and assume G acts on $\mathcal{C}$ by permutations of the set $\mathcal{C}$ in such a way that

$$r(g \cdot \alpha) = g \cdot r(\alpha) \quad s(g \cdot \alpha) = g \cdot s(\alpha) \quad \text{for all } g \in G, \alpha \in \mathcal{C}.$$

A map $\varphi\colon G \times \mathcal{C} \to G$ is called a *category cocycle*, and $(\mathcal{C}, G, \varphi)$ is called a *category system*, if for all $g, h \in G$, $v \in \mathcal{C}^0$, and $\alpha, \beta \in \mathcal{C}$ such that $s(\alpha) = r(\beta)$, we have

1. $\varphi(g, v) = g$,
2. $\varphi(g, \alpha) \cdot s(\alpha) = g \cdot s(\alpha)$,
3. $g \cdot (\alpha\beta) = (g \cdot \alpha)(\varphi(g, \alpha) \cdot \beta)$,
4. $\varphi(g, \alpha\beta) = \varphi\big(\varphi(g, \alpha), \beta\big)$,
5. $\varphi(gh, \alpha) = \varphi(g, h \cdot \alpha)\, \varphi(h, \alpha)$.

Note: the "trivial" choice $\varphi(g, \alpha) := g$ for all $g \in G, \alpha \in \mathcal{C}$, is the only case where we are guaranteed that G acts on $\mathcal{C}$ by automorphisms of the category.

Proposition

Let $(\mathcal{C}, G, \varphi)$ be a category system.
Set $\mathcal{D} = \mathcal{C} \times G$, $\mathcal{D}^0 = \mathcal{C}^0 \times \{1\}$, and define $r, s\colon \mathcal{D} \to \mathcal{D}^0$ by

$$r(\alpha, g) = \big(r(\alpha), 1\big) \quad \textit{and} \quad s(\alpha, g) = \big(g^{-1} \cdot s(\alpha), 1\big).$$

For $(\alpha, g), (\beta, h) \in \mathcal{D}$ with $s(\alpha, g) = r(\beta, h)$, we have $s(\alpha) = r(g \cdot \beta)$, so we can define

$$(\alpha, g)(\beta, h) = \big(\alpha(g \cdot \beta), \varphi(g, \beta)h\big).$$

Then $\mathcal{D}$ is a small category.

For a more general result, see Brin (2005).
We denote $\mathcal{D}$ by $\mathcal{C} \rtimes^{\varphi} G$, and call it the *Zappa-Szép product* of $(\mathcal{C}, G, \varphi)$.

Proposition

Let $(\mathcal{C}, G, \varphi)$ be a category system and assume $\mathcal{C}$ is left cancellative.

Then $\mathcal{D} = \mathcal{C} \rtimes^{\varphi} G$ is also left cancellative.

Further, if $\mathcal{C}$ is finitely aligned, then so is $\mathcal{D}$, and for all $(\alpha, g), (\beta, h) \in \mathcal{D}$ we have

$$(\alpha, g) \vee (\beta, h) = (\alpha \vee \beta) \times \{1\}.$$

In particular, if $\mathcal{C}$ is singly aligned, then so is $\mathcal{D}$.

For the case where $\mathcal{C}$ is a monoid which is right LCM (i.e., a singly aligned left cancellative monoid) and G is a monoid, see Brownlowe, Rammage, Robertson and Whittaker (2014), (and Lawson (2008)).

Definition

By a *representation* of $(\mathcal{C}, G, \varphi)$ in a C^*-algebra B we mean a pair (T, U), where $T\colon \mathcal{C} \to B$ is a nondegenerate representation of $\mathcal{C}$ and $U\colon G \to M(B)$ is a unitary homomorphism such that

$$U_g T_\alpha = T_{g\alpha} U_{\varphi(g,\alpha)} \quad \text{for all } g \in G, \alpha \in \mathcal{C}.$$

Theorem

- *If (T, U) is a representation of $(\mathcal{C}, G, \varphi)$ in B, then the map $T \times U\colon \mathcal{D} \to B$, defined by $(T \times U)_{(\alpha,g)} = T_\alpha U_g$, is a nondegenerate representation of $\mathcal{D} = \mathcal{C} \rtimes^\varphi G$.*
- *Conversely, if $R\colon \mathcal{D} \to B$ is a nondegenerate representation of $\mathcal{D}$, then the series $\sum_{v \in \mathcal{C}^0} R_{(v,g)}$ converges strictly in $M(B)$, and the pair (T^R, U^R), defined by*

$$T^R_\alpha = R_{(\alpha,1)}\,,\; U^R_g = \sum_{v \in \mathcal{C}^0} R_{(v,g)},$$

  is a representation of $(\mathcal{C}, G, \varphi)$.
- *The maps $(T, U) \to T \times U$ and $R \to (T^R, U^R)$ are inverses of each other.*

Corollary

$\mathcal{T}(\mathcal{D})$ is the universal C^*-algebra for representations of $(\mathcal{C}, G, \varphi)$.

Indeed, there exists a representation $(t^{\mathcal{D}}, u^{\mathcal{D}})$ of $(\mathcal{C}, G, \varphi)$ in $\mathcal{T}(\mathcal{D})$ with the following universal property:

If (T, U) is a representation of $(\mathcal{C}, G, \varphi)$ in a C^*-algebra B, then there is a unique homomorphism $\phi : \mathcal{T}(\mathcal{D}) \to B$ such that

$$T = \phi \circ t^{\mathcal{D}} \quad \textit{and} \quad U = \phi \circ u^{\mathcal{D}}.$$

Proof: Let $r^{\mathcal{D}} \colon \mathcal{D} \to \mathcal{T}(\mathcal{D})$ denote the universal representation of $\mathcal{D}$, and write $r^{\mathcal{D}} = t^{\mathcal{D}} \times u^{\mathcal{D}}$. Then $(t^{\mathcal{D}}, u^{\mathcal{D}})$ has the desired property. Indeed, given (T, U), it suffices to set $\phi := \phi_{T \times U}$.

Definition

A representation (T, U) of $(\mathcal{C}, G, \varphi)$ in a C^*-algebra B is called *covariant* if $T\colon \mathcal{C} \to B$ is covariant.

Proposition

A representation (T, U) of $(\mathcal{C}, G, \varphi)$ is covariant if and only if $T \times U$ is a covariant representation of $\mathcal{D}$.

Corollary

$\mathcal{O}(\mathcal{D})$ is the universal C^*-algebra for covariant representations of $(\mathcal{C}, G, \varphi)$.

[We can let $\tilde{r}^{\mathcal{D}}\colon \mathcal{D} \to \mathcal{O}(\mathcal{D})$ denote the universal covariant representation of $\mathcal{D}$, and proceed as in the Toeplitz case.]

Exel-Pardo systems II

Let (E, G, φ) be an Exel-Pardo system, and let E^* denote the category of all finite paths in E.

Proposition (Exel-Pardo)

(E, G, φ) extends uniquely to a category system (E^*, G, φ).

Let $g \in G$. Set $\varphi(g, v) := g$ for all $v \in E^0 = (E^*)^0$.

For $\alpha \in E^1$, set $g_{|\alpha} := \varphi(g, \alpha)$.

Consider $\gamma = \alpha_1 \alpha_2 \cdots \alpha_n \in E^*$, $n \geq 1$.

Then define $g_0^\gamma, g_1^\gamma, \ldots, g_n^\gamma \in G$ recursively by

$$\begin{aligned} g_0^\gamma &:= g, \\ g_k^\gamma &:= (g_{k-1}^\gamma)_{|\alpha_k} = (\cdots(g_{|\alpha_1})_{|\alpha_2} \cdots)_{|\alpha_k}, \; 1 \leq k \leq n. \end{aligned}$$

Now define

$$\begin{aligned} g \cdot \gamma &:= (g_0^\gamma \cdot \alpha_1)(g_1^\gamma \cdot \alpha_2) \cdots (g_{n-1}^\gamma \cdot \alpha_n), \\ \varphi(g, \gamma) &:= g_n^\gamma = (\cdots(g_{|\alpha_1})_{|\alpha_2} \cdots)_{|\alpha_n}. \end{aligned}$$

Since E^* is a singly aligned LCSC, we get:

Corollary

Let (E, G, φ) be an Exel-Pardo system. Then the Zappa-Szép product $E^* \bowtie^{\varphi} G$ is a singly aligned LCSC.

Theorem

Assume E is row-finite.

Then there is a bijective correspondence between

- *representations (resp. covariant repr.) of (E, G, φ) in a C^*-algebra B, and*
- *nondegenerate representations (resp. nondegenerate covariant repr.) of $E^* \bowtie^{\varphi} G$ in B*

It follows that

$$\mathcal{T}(E, G, \varphi) \simeq \mathcal{T}(E^* \bowtie^{\varphi} G)\,,\ \mathcal{O}(E, G, \varphi) \simeq \mathcal{O}(E^* \bowtie^{\varphi} G).$$

The proof goes as follows:

Let (P, S, U) be a representation of (E, G, φ) in B.

- (P, S) determines a nondegenerate representation $T\colon E^* \to B$.
- (T, U) is then a representation of (E^*, G, φ) in B.

So $R := T \times U$ is a nondegenerate representation of $\mathcal{D}$ in B. If (P, S, U) is covariant, then T is covariant, so R is too.

Conversely, let $R\colon \mathcal{D} \to B$ be a nondegenerate representation.

- Let $(T, U) := (T^R, U^R)$ be the associated representation of (E^*, G, φ) in B.
- Let (P, S) be the Toeplitz E-family in B naturally associated to T.

Then (P, S, U) is a representation of (E, G, φ) in B. If R is covariant then so is (P, S, U).

More on finitely aligned LCSC

Let $\mathcal{C}$ denote a finitely aligned LCSC. One may also construct $\mathcal{T}(\mathcal{C})$ and $\mathcal{O}(\mathcal{C})$ by adapting Donsig and Milan's approach (2014) to Spielberg's categories of paths.

For $\alpha \in \mathcal{C}$, the map $\tau_\alpha \colon s(\alpha)\mathcal{C} \to \alpha\mathcal{C}$ given by

$$\tau_\alpha(\beta) = \alpha\beta \quad \text{for all} \quad \beta \in s(\alpha)\mathcal{C}$$

is a bijection, with inverse $\sigma_\alpha \colon \alpha\mathcal{C} \to s(\alpha)\mathcal{C}$ given by

$$\sigma_\alpha(\alpha\beta) = \beta \quad \text{for all } \beta \in s(\alpha)\mathcal{C}.$$

Letting $I(\mathcal{C})$ denote the symmetric inverse semigroup of $\mathcal{C}$, consisting of all partial bijections of $\mathcal{C}$, we let $\mathrm{ZM}(\mathcal{C})$ be the inverse subsemigroup of $I(\mathcal{C})$ generated by

$$\{\tau_\alpha, \sigma_\alpha\}_{\alpha \in \mathcal{C}} \cup \{\mathrm{id}_\varnothing\}.$$

$\mathrm{ZM}(\mathcal{C})$ may be described by using *zigzags* on $\mathcal{C}$:

Let $\mathcal{Z}_\mathcal{C}$ denote the set of even tuples of the form

$$\zeta = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n),$$

where $n \geq 1$, $\alpha_i, \beta_i \in \mathcal{C}$, $r(\alpha_i) = r(\beta_i)$ for $i = 1, \ldots, n$, and $s(\beta_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n-1$.

Define $\varphi_\zeta \in \mathrm{ZM}(\mathcal{C})$ by

$$\varphi_\zeta = \sigma_{\alpha_1}\, \tau_{\beta_1}\, \cdots\, \sigma_{\alpha_n}\, \tau_{\beta_n}.$$

Note that

$$\varphi_{(\alpha, r(\alpha))} = \sigma_\alpha\,,\ \varphi_{(r(\beta), \beta)} = \tau_\beta.$$

Moreover, we have

$$\mathrm{ZM}(\mathcal{C}) = \{\varphi_\zeta : \zeta \in \mathcal{Z}_\mathcal{C}\} \cup \{\mathrm{id}_\varnothing\}$$

and

$$E(\mathrm{ZM}(\mathcal{C})) = \left\{\mathrm{id}_{\mathrm{dom}(\varphi_\zeta)} : \zeta \in \mathcal{Z}_\mathcal{C}\right\} \cup \{\mathrm{id}_\varnothing\}.$$

Let $C^*(\mathrm{ZM}(\mathcal{C}))$ denote the C^*-algebra which is universal for (0-preserving) representations of $\mathrm{ZM}(\mathcal{C})$.

One can show that there is a bijective correspondence between representations of $\mathcal{C}$ and *finitely join-preserving* representations of $\mathrm{ZM}(\mathcal{C})$.

The Toeplitz algebra $\mathcal{T}(\mathcal{C})$ may therefore be defined as the quotient of $C^*(\mathrm{ZM}(\mathcal{C}))$ which is universal for *finitely join-preserving* representations of $\mathrm{ZM}(\mathcal{C})$.

Moreover, covariant representations of $\mathcal{C}$ correspond to *cover-to-join* ($=$ tight) representations of $\mathrm{ZM}(\mathcal{C})$.

So $\mathcal{O}(\mathcal{C})$ may be defined as the quotient of $C^*(\mathrm{ZM}(\mathcal{C}))$ which is universal for cover-to-join ($=$ tight) representations of $\mathrm{ZM}(\mathcal{C})$.

One may also introduce the regular Toeplitz algebra $\mathcal{T}_\ell(\mathcal{C})$ and the Li C^*-algebra $C^*_{\mathrm{Li}}(\mathcal{C})$.