A perspective on non-commutative frame theory

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Facets of irreversibility:
inverse semigroups, groupoids and operator algebras

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The talk is based on the paper:

Ganna Kudryavtseva, Mark V. Lawson,
A perspective on non-commutative frame theory, Adv. Math. **311** (2017), 378 – 468.

which extends and unifies ideas of the papers:

- Pedro Resende, *Etale groupoids and their quantales*, *Adv. Math.* **208** (2007), 117 – 170.
- Mark V. Lawson, Daniel H. Lenz, *Pseudogroups and their étale groupoids*, *Adv. Math.* **244** (2013), 147 – 209.

Overview: pseudogroups as non-commutative frames

- Psedogroup - inverse semigroup S such that $E(S)$ is a frame and any compatible set of elements in S has a join in S.
- Prototypical example: pseudogroup of homeomorphisms between open sets of a topological space.
- Aim: extend classical dualities (P. Johnstone, Stone spaces) from frames to pseudogroups. The role of locales (resp. topological spaces) will be taken over by étale localic (resp. topological) groupoids.
- Pedro Resende 2007 (equivalence between pseudogroups and groupoids, mediated by quantales, at the level of objects).
- Mark Lawson and Daniel Lenz 2013 (equivalence between pseudogroups and groupoids, objects + morphisms).
- GK and Mark Lawson 2017 (all above equivalences made functorial, four natural types of morphisms considered).

Overview: groupoids replaced by categories

- "Non-commutative frame": does one really need the structure of an inverse semigroup? In particular, is the presence of inverses of crucial importance?
- Due to presence of inverses, one gets a groupoid at the topological side of dualities. We modify the constructions aimed to get a category, rather than a groupoid.
- This generalizes and simplifies at the same time! No inverses - less structure - easier constructions.
- Have étale caterogies been considered in relation to non-selfadjoint operator algebras?
- What is the appropriate replacement of inverse semigroups? Semigroups related to inverse semigroups in a similar way to as groupoids are related to categories.
- Have such semigroups been considered in semigroup theory literature?

Étale categories in the study of subalgebras of combinatorial C^*-algebras

For (certain) subalgebras of AF C^* algebras Power (1990) showed that they are classified by "topological binary relations" (so principal étale categories impliciely mentioned!) which are contained in "topological equivalence relations" (=principal étale groupoids).
A. Hopenwasser, J. R. Peters, and S. C. Power in "Subalgebras of graph C* algebras" considered nest subalgebras of graph C^* algebras associated to a finite graph with a total order.
Research direction:

- Connect non-selfadjoint subalgebras of groupoid C^*-algebras (e.g. AF-algebras or "combinatorial algebras", e.g., graph algebras,...) with respective étale subcategories of étale groupoids, and also with underlying restriction subsemigroups of inverse semigroups.
- Define the algebra of restriction semigroup (modification of groupoid approach or of Duncan's and Paterson's approach)

Restriction semigroups: idea

- Restriction semigroups (also known as weakly E-ample semigroups) are non-regular generalizations of inverse semigroups. These are semigroups with two unary operations $a \mapsto a^+$ and $a \mapsto a^*$ which mimic the operations $a \mapsto a^{-1}a$ (the domain idempotent) and $a \mapsto aa^{-1}$ (the range idempotent) in an inverse semigroup.
- Some examples: $O(X)$; reflexive and transitive relations and bisections of their powersets.

A key insight:

(Ehresmann-Nambooripad-Schein) The category of inverse semigroups is isomorphic to the category of inductive groupoids.
(Mark Lawson) The category of restriction semigroups is isomorphic to the category of inductive categories.

Historical note

C. Hollings, From right PP monoids to restriction semigroups: a survey.

- Restriction semigroups (terminology from restriction categories by Cockett and Lack), formely weakly E-ample semigroups were first considered (according to the above survey) by A. El-Qallali in 1980. ("York school", John Fountain's earlier related works).
- Ehresmann semigroups (more general than restriction semigroups and important for this talk) were introduced by Mark Lawson in 1991.

Restriction semigroups: definition

A restriction semigroup is an algebra $(S; \cdot, ^*, ^+)$ of type $(2,1,1)$ such that $(S, \cdot)$ is a semigroup and the following axioms hold

$$xx^* = x, x^*y^* = y^*x^*, (xy^*)^* = x^*y^*, x^*y = y(xy)^*; \qquad (1)$$

$$x^+x = x, x^+y^+ = y^+x^+, (x^+y)^+ = x^+y^+, xy^+ = (xy)^+x; \qquad (2)$$

$$(x^+)^* = x^+, (x^*)^+ = x^*. \qquad (3)$$

Semilattice of projections of S:

$$E = \{x^* \colon x \in S\} = \{x^+ \colon x \in S\}.$$

Natural partial order

$$a \leq b \Leftrightarrow a = eb \text{ for some } e \in E \Leftrightarrow a = bf \text{ for some } f \in E.$$

Complete restriction monoids

- S - restriction semigroup, $a, b \in S$.
- a and b are compatible if $ab^* = ba^*$ and $b^+a = a^+b$. Write $a \sim b$.
- S is called a complete restriction monoid if E is a frame and joins of compatible families of elements exist in S.
- A pseudogroup is a complete restriction monoid which is admits a structure of an inverse semigroup.
- Key example: Let $C = (C_1, C_0)$ be an étale localic (or topological) category. Then the set of all its local bisections forms a complete restriction monoid.
- Corollary: Let $C = (C_1, C_0)$ be an étale localic (or topological) groupoid. Then the set of all its local bisections forms a pseudogroup.

Quantales

A quantale $(Q, \leq, \cdot)$ is a sup-lattice $(Q, \leq)$ equipped with a binary multiplication operation $\cdot$ such that multiplication distributes over arbitrary suprema:

$$a(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(ab_i) \text{ and } (\bigvee_{i\in I} b_i)a = \bigvee_{i\in I}(b_i a).$$

A quantale is unital if there is a multiplicative unit e and involutive, if there is an involution $*$ on Q which is a sup-lattice endomorphism.

A quantal frame is a quantale which is also a frame.

Quantale from a localic category

Let $C = (C_1, C_0)$ be a localic (or topological) category and assume that the multiplication map $m\colon C_1 \times_{C_0} C_1 \to C_1$ is semiopen. Then there is the direct image map

$$m_!\colon O(C_1 \times_{C_0} C_1) \to O(C_1)$$

and

$$m_! q\colon O(C_1) \otimes O(C_1) \to O(C_1)$$

is a "globalization" of multiplication from points to open sets (and similarly for a topological category).

- $O(C_1)$ is a quantale.

Quantale from a localic category

Let $C = (C_1, C_0; d, r, u, m)$ be a localic category with maps d, r, u open and m semiopen.

- $a^* = u_! d_!(a)$, $a \in O(C_1)$.
- $a^+ = u_! r_!(a)$, $a \in O(C_1)$.
- $e = u_!(1_{O(C_0)})$.

Theorem (Correspondence Theorem)

1. $(O(C_1), e, ^+, ^*)$ is a multiplicative Ehresmann quantal frame.
2. Any multiplicative Ehresmann quantal frame arises in this way.

$(O(C_1), ^*, ^+)$ is an Ehresmann semigroup - generalizations of restriction semigroups, equations $x^*y = y(xy)^*$ and $xy^+ = (xy)^+x$ are not required.

Partial isometries

- Q – an Ehresmann quantal frame
- $a \in Q$
- a is a partial isometry if $b \leq a$ implies that $b = af = ga$ for some $f, g \leq e$
- Notation: $\mathcal{PI}(Q)$.
- Partial isometries are abstract analogues of local bisections.

Example

X a non-empty set, $A \subseteq X \times X$ a transitive and reflexive relation. The partial isometries of the Ehresmann quantal frame $\mathcal{P}(A)$ are precisely partial bijections.

Étale Correspondence Theorem

A localic category $C = (C_1, C_0)$ is étale if u, m are open and d, r are local homeomorphisms.

An Ehresmann quantal frame Q is a restriction quantal frame if every element is a join of partial isometries and partial isometries are closed under multiplication.

Theorem (Étale Correspondence Theorem)

There is an equivelence between restriction quantal frames and étale localic categories.

This extends and is inspired by the correspondence between inverse quantal frames and étale localic groupoids due to Pedro Resende.

Restriction quantal frames and complete restriction monoids

- An Ehresmann quantal frame Q is a restriction quantal frame if every element is a join of partial isometries and partial isometries are closed under multiplication.
- An Ehresmann quantal frame Q, restricted to partial izometries, is a complete restriction monoid.
- If S is a complete restriction monoid, then $\mathcal{L}^{\vee}(S)$, the set of $\vee$-closed order ideals of S, is a restriction quantal frame.

Theorem

The following categories are equivalent:

- Complete restriction monoids.
- Restriction quantal frames.
- Étale localic categories.

Morphisms

A morphism $\varphi : Q_1 \to Q_2$ between Ehresmann quantal frames is a quantale map that is also a map of Ehresmann monoids (preserves both $*$ and $+$).

Four types of morphisms between Ehresmann quantal frames:

- type 1: morphisms;
- type 2: proper morphisms (unital morphism=preserves the top element);
- type 3: $\wedge$-morphisms (preserves non-empty finite meets);
- type 4: proper $\wedge$-morphisms (preserves all finite meets).

Morphisms between respective quantal localic categories are defined as the above morphisms but in the opposite direction. Only type 4 morphisms give rise to functors between categories!

The adjunction

Theorem

There is an adjunction between the category of étale localic categories and the category of étale topological categories.

This adjunction extends the classical adjunction between locales and topological spaces.

Corollary

There is a dual adjunction between the category restriction quantal frames and the category of étale topological categories.

This adjunction extends the classical dual adjunction between frames and topological spaces.

The adjunction

Lawson and Lenz constructed an adjunction between pseudogroups and étale topological groupoids. Our results show that it be decomposed into the following steps:

$S -$ pseudogroup;

$\mathcal{L}^{\vee}(S) -$ the enveloping inverse quantal frame of S;

$\mathcal{G}(\mathcal{L}^{\vee}(S)) -$ the corresponding étale localic groupoid;

$\mathrm{pt}(\mathcal{G}(\mathcal{L}^{\vee}(S))) -$ the projection to an étale topological groupoid.

The adjunction

In the reverse direction:

G – a topological étale groupoid;

$\Omega(G)$ – a localic étale groupoid;

$\mathcal{Q}(\Omega(G))$ – the inverse quantal frame of $\Omega(G)$;

$\mathcal{PI}(\mathcal{Q}(\Omega(G)))$ – the pseudogroup of partial isometries.

Relational covering morphisms

Let $C = (C_1, C_0)$ and $D = (D_1, D_0)$ be étale topological categories. A relational covering morphism from C to D as a pair $f = (f_1, f_0)$, where

- $f_0 : C_0 \to D_0$ is a continuous map,
- $f_1 : C_1 \to \mathcal{P}(D_1)$ is a function,

and the following axioms are satisfied:

(RM1) If $b \in f_1(a)$ where $a \in C_1$ then $d(b) = f_0 d(a)$ and $r(b) = f_0 r(a)$.

(RM2) If $(a, b) \in C_1 \times_{C_0} C_1$ and $(c, d) \in D_1 \times_{D_0} D_1$ are such that $c \in f_1(a)$ and $d \in f_1(b)$ then $cd \in f_1(ab)$.

(RM3) If $d(a) = d(b)$ (or $r(a) = r(b)$) where $a, b \in C_1$ and $f_1(a) \cap f_1(b) \neq \varnothing$ then $a = b$.

(RM4) If $p = f_0(q)$ and $d(s) = p$ (resp. $r(s) = p$) where $q \in C_0$ and $s \in D_1$ then there is $t \in C_1$ such that $d(t) = q$ (resp. $r(t) = q$) and $s \in f_1(t)$.

(RM5) For any $A \in O(D_1)$: $f_1^{-1}(A) = \{x \in C_1 \colon f_1(x) \cap A \neq \varnothing\} \in O(C_1)$.

(RM6) $uf_0(t) \in f_1 u(t)$ for any $t \in C_0$.

(RM2) - weak form of preservation of multiplication;
(RM3) and (RM4) – f_1 is star-injective and star-surjective;
(RM5) – f_1 is a lower-semicontinuous relation.

From quantale to topological morphisms

Let $C = (C_1, C_0)$ and $D = (D_1, D_0)$ be étale localic categories and $f_1^* \colon \mathcal{O}(D) \to \mathcal{O}(C)$ a morphism of restriction quantal frames.

Theorem

- If f_1^* is of type 1 then $\mathrm{Pt}(f_1)$ is a relational covering morphism.
- If f_1^* is of type 2 (=proper=unital) then $\mathrm{Pt}(f_1)$ is at least single-valued relational covering morphism.
- If f_1^* is of type 3 (preserves non-empty finite meets) then $\mathrm{Pt}(f_1)$ is at most single valued relational covering morphism.
- If f_1^* is of type 4 (preserves finite meets) then $\mathrm{Pt}(f_1)$ is a single-valued relational covering morphism. This recovers continuous covering functors of Lawson and Lenz.

Sober-spatial equivalences

- Let $C = (C_1, C_0)$ be an étale localic category. Then the locale C_1 is spatial iff the locale C_0 is spatial. If these hold C is called spatial.
- Let $C = (C_1, C_0)$ be an étale topological category. Then the space C_1 is sober iff the space C_0 is sober. If these hold C is called sober.

Theorem

The category of spatial étale localic categories is equivalent to the category of sober étale topological categories.

Spectral, coherent and Boolean categories

- An étale localic category $C = (C_1, C_0)$ is called coherent (resp. strongly coherent) if the locale C_0 (resp. C_1) is coherent.
- There is an equivalence of categories between coherent (resp. strongly coherent) étale localic categories and distributive restriction semigroups (resp. distributive restriction $\wedge$-semigroups).
- An étale topological category $C = (C_1, C_0)$ is called spectral (resp. strongly spectral) if the space C_0 (resp. C_1) is spectral.
- An étale topological category $C = (C_1, C_0)$ is called Boolean (resp. strongly Boolean) if the space C_0 (resp. C_1) is Boolean (locally compact).

The topological duality theorem

Topological duality theorems

- The category of distributive restriction semigroups (resp. $\wedge$-semigroups) is dual to the category of spectral (resp. strongly spectral) étale topological categories.
- The category of Boolean restriction semigroups (resp. $\wedge$-semigroups) is dual to the category of Boolean (resp. strongly Boolean) étale topological categories.

Remark. All results above have corollaries with

- restriction semigroups replaced with inverse semigroups and categories replaced by groupoids.

Summary of morphisms

Type	Inv. sem. morphism	$\wedge$-inv. sem. morphism
1	morphism	morphism (no $\wedge$-preservation)
2	proper morphism	proper morphism
3	weakly $\wedge$-preserving	$\wedge$-preserving
4	proper and weakly $\wedge$-preserving	proper and $\wedge$-preserving

Summary of topological dualities (inverse semigroups)

Algebraic object	Topological étale groupoid $C = (C_1, C_0)$
Spatial pseudogroup	C_0 or C_1 (and then both) sober
Coherent pseudogroup	C_0 coherent
Strongly coherent pseudogroup	C_1 coherent
Distributive inv. sem.	C_0 – spectral
Distributive $\wedge$ inv. sem.	C_1 (and thus also C_0) spectral
Boolean inv. sem.	C_0 – Boolean
Boolean $\wedge$ inv. sem.	C_1 (and thus also C_0) Boolean

The results hold also in a wider setting: if "inverse semigroup" is replaced by "restriction semigoup" and "groupoid" by "category".