Groupoids and higher-rank graphs

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(Ongoing work with Toke Meier Carlsen)

Higher-rank graphs

Definition (Kumjian–Pask)

A *higher-rank graph* or *k-graph* is a countable small category $\Lambda := (\mathrm{obj}(\Lambda), \mathrm{mor}(\Lambda), r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ with $d(\mu) = m$ and $d(\nu) = n$ such that $\lambda = \mu\nu$. We then write $\lambda(0, m) := \mu$ and $\lambda(m, m + n) = \nu$.

Definition (Kumjian–Pask)

A k-graph is called *row-finite* if the set $v\Lambda^m := \{\lambda \in \Lambda : d(\lambda) = m, r(\lambda) = v\}$ is finite for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$. If each of the sets $v\Lambda^m$ is nonempty, then Λ has *no sources*.

Examples of k-graphs

Example (Directed graphs)

If Λ is a 1-graph, then $(d^{-1}(0), d^{-1}(1), r, s)$ is a directed graph. Conversely, if $E = (E^0, E^1, r, s)$ is a directed graph, then $E^* := \bigcup_{n \geq 0} E^n$, the collection of finite paths, may be viewed as small category with range and source given by $r(e_n \ldots e_1) := r(e_n)$ and $s(e_n \ldots e_1) := s(e_1)$. Taking $d : E^* \to \mathbb{N}$ to be the length function i.e. $d(e_n \ldots e_1) = n$, we have that (E^*, d) is a 1-graph.

Example (Kumjian–Pask)

For $k \in \mathbb{N}$, define $\Omega_{k,\infty} := \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q\}$. This is a k-graph with $\Omega_{k,\infty}^0 = \mathbb{N}^k$, $r(p,q) = p$, $s(p,q) = q$, $d(p,q) = q - p$ and composition defined by $(p,q)(q,r) = (p,r)$.

Definition (Kumjian–Pask)

Let Λ be a row-finite k-graph with no sources. A *Cuntz–Krieger* Λ-family is a collection $\{S_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra A satisfying

- $\{S_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
- $S_\lambda S_\mu = S_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
- $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
- for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ we have $S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*$.

Theorem (Kumjian–Pask)

Given a row-finite k-graph Λ with no sources, there is a C^*-algebra $C^*(\Lambda)$ generated by a Cuntz–Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the following sense: for any Cuntz–Kriger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra A, there is a unique homomorphism $\pi_t : C^*(\Lambda) \to A$ such that $\pi_S(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

This C^*-algebra can be written $C^*(\Lambda) = \overline{\text{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}$. The *diagonal subalgebra* of $C^*(\Lambda)$ is given by $\mathcal{D}(\Lambda) := \overline{\text{span}}\{s_\mu s_\mu^* : \mu \in \Lambda\}$.

For a k_1-graph Λ_1 and a k_2-graph Λ_2, we say that an isomorphism $\phi : C^*(\Lambda_1) \to C^*(\Lambda_2)$ is *diagonal-preserving* if $\phi(\mathcal{D}(\Lambda_1)) = \mathcal{D}(\Lambda_2)$.

Kumjian–Pask families

For each $\lambda \in \Lambda \setminus \Lambda^0$, we define a *ghost path* by a formal symbol λ^*. For $v \in \Lambda^0$, we define $v^* := v$, and extend r and s to the collection of ghost paths by setting $r(\lambda^*) := s(\lambda)$ and $s(\lambda) := r(\lambda^*)$. Composition of ghost paths is defined by $\lambda^* \mu^* := (\mu\lambda)^*$.

Definition (Aranda Pino–Clark–an Huef–Raeburn)

Let Λ be a row-finite k-graph with no sources, and R a commutative ring with identity. A *Kumjian–Pask* Λ-family $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ in an R-algebra A consists of a function $S : \Lambda \cup \{\mu^* : \mu \in \Lambda \setminus \Lambda^0\} \to A$ satisfying

- $\{S_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal idempotents;
- $S_\lambda S_\mu = S_{\lambda\mu}$ and $S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}$ whenever $s(\lambda) = r(\mu)$;
- $S_{\lambda^*} S_\lambda = S_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
- for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ we have $S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_{\lambda^*}$.

Theorem (Aranda Pino–Clark–an Huef–Raeburn)

Given a row-finite k-graph with no sources and a commutative ring R with identity, there is an R-algebra $\mathrm{KP}(\Lambda)$ *generated by a Kumjian–Pask* Λ*-family* $\{s_\lambda, s_{\mu^*} : \lambda, \mu \in \Lambda\}$ *which is universal in the following sense: for any Kumjian–Pask* Λ*-family* $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ *in an R-algebra A, there is a unique R-algebra homomorphism* $\pi_S : \mathrm{KP}(\Lambda) \to A$ *such that* $\pi_S(s_\lambda) = S_\lambda$ *and* $\pi_S(s_{\mu^*}) = S_{\mu^*}$ *for all* $\lambda, \mu \in \Lambda$*.*

This R-algebra can be written $KP(\Lambda) = \mathrm{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}$. The *diagonal subalgebra* of $\mathrm{KP}(\Lambda)$ is given by $D_R(\Lambda) := \mathrm{span}_R\{s_\mu s_{\mu^*} : \mu \in \Lambda\}$.

For a k_1-graph Λ_1 and a k_2-graph Λ_2, we say that a ring-isomorphism $\phi : \mathrm{KP}(\Lambda_1) \to \mathrm{KP}(\Lambda_2)$ is *diagonal-preserving* if $\phi(D_(\Lambda_1)) = D(\Lambda_2)$.

The infinite path space

Definition (Kumjian–Pask)

Let Λ be a row-finite k-graph with no sources. An *infinite path* is a degree-preserving functor $x : \Omega_{k,\infty} \to \Lambda$. The collection of infinite paths is denoted by Λ^∞. For $v \in \Lambda^0$, write $v\Lambda^\infty := \{x \in \Lambda^\infty : x(0) = v\}$. For each $p \in \mathbb{N}^k$, define $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m, n) = x(m + p, n + p)$ for $x \in \Lambda^\infty$ and $(m, n) \in \Omega_{k,\infty}$.

Lemma (Kumjian–Pask)

For $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $x(0) = s(\lambda)$, there exists $\lambda x \in \Lambda^\infty$ such that $x = \sigma^{d(\lambda)}(\lambda x)$ and $\lambda = (\lambda x)(0, d(\lambda))$.

Define $Z(\lambda) := \{\lambda x : x \in \Lambda^\infty, x(0) = s(\lambda)\}$. The sets $Z(\lambda)$ form a basis for a locally compact Hausdorff topology on Λ^∞.

The path groupoid

Definition (Kumjian–Pask)

The *path groupoid* $\mathcal{G}_\Lambda$ is given by

$$\mathcal{G}_\Lambda := \{(x, p-q, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : p, q \in \mathbb{N}^k, \sigma^p(x) = \sigma^q(y)\},$$

with partially-defined product $(x, m, y)(y, n, z) \mapsto (x, m+n, z)$, inverse operation $(x, m, y) \mapsto (y, -m, x)$, and range and source maps $r(x, m, y) := x$ and $s(x, m, y) := y$. We identify the infinite path space Λ^∞ with the unit space $\mathcal{G}_\Lambda^0$ via the map $x \to (x, 0, x)$.

For $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$, write

$$Z(\lambda *_s \mu) := \{(\lambda z, d(\lambda) - d(\mu), \mu z) : z \in s(\lambda)\Lambda^\infty\},$$

The sets $Z(\lambda *_s \mu)$ form a basis for a topology that makes $\mathcal{G}_\Lambda$ a locally compact, Hausdorff, étale and ample groupoid.

For a row-finite k-graph Λ with no sources, there is an isomorphism $\pi : C^*(\Lambda) \to C^*(\mathcal{G}_\Lambda)$ satisfying

$$\pi(s_\lambda) = 1_{Z(\lambda *_s s(\lambda))}$$

for all $\lambda \in \Lambda$. Moreover, $\pi(\mathcal{D}(\Lambda)) \cong C_0(\mathcal{G}_\Lambda^0) \subseteq C^*(\mathcal{G}_\Lambda)$,

For a row-finite k-graph Λ with no sources and a commutative ring R with identity, there is an isomorphism $\pi_T : \mathrm{KP}(\Lambda) \to A_R(G_\Lambda)$ such that

$$\pi_T(s_\lambda) = 1_{Z(\lambda *_s s(\lambda))} \text{ and } \pi_T(s_{\lambda^*}) = 1_{Z(s(\lambda) *_s \lambda)}$$

for $\lambda \in \Lambda$. Moreover, $\pi_T(D_R(\Lambda)) \cong A_R(\mathcal{G}_\Lambda^0) \subseteq A_R(\mathcal{G}_\Lambda)$.

Diagonal-preserving isomorphisms of k-graph C^*-algebras and Kumjian–Pask algebras

For a k-graph Λ, there is a cocycle $c_\Lambda : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ given by $c_\Lambda(x, n, y) = n$, a gauge action $\gamma^\Lambda : \mathbb{T}^k \to \operatorname{aut} C^*(\Lambda)$ given by $\gamma_z^\Lambda(s_\lambda) = z^{d(\lambda)} s_\lambda$ for $z \in \mathbb{T}^k$ and a $\mathbb{Z}^k$-grading $\mathrm{KP}(\Lambda) = \bigoplus_{n \in \mathbb{Z}^k} \mathrm{KP}(\Lambda)_n$ where $\mathrm{KP}(\Lambda)_n = \operatorname{span}_R\{s_\mu s_\lambda^* : \mu, \nu \in \Lambda,\ d(\mu) - d(\nu) = n\}$.

Theorem (Carlsen–Ruiz–Sims–Tomforde, Carlsen–R)

Let Λ_1 and Λ_2 be k-graphs and let R be an integral domain with identity. TFAE

- *There is an isomorphism $\Phi : \mathcal{G}_{\Lambda_1} \to \mathcal{G}_{\Lambda_2}$ satisfying $c_{\Lambda_2} \circ \Phi = c_{\Lambda_1}$.*
- *There is a diagonal-preserving $*$-isomorphism $\Psi : C^*(\Lambda_1) \to C^*(\Lambda_2)$ satisfying $\gamma_z^{\Lambda_2} \circ \Psi = \Psi \circ \gamma_z^{\Lambda_1}$ for $z \in \mathbb{T}^k$.*
- *There is a diagonal-preserving ring-isomorphism $\theta : \mathrm{KP}(\Lambda_1) \to \mathrm{KP}(\Lambda_2)$ satisfying $\theta(\mathrm{KP}(\Lambda_1)_n) = \mathrm{KP}(\Lambda_2)_n$ for $n \in \mathbb{Z}^k$.*

Definition (Carlsen–R, Matsumoto)

We say that k-graphs Λ_1 and Λ_2 are *eventually one-sided conjugate* if there is a homeomorphism $h : \Lambda_1^\infty \to \Lambda_2^\infty$ and continuous maps $f_m : \Lambda_1^\infty \to \mathbb{N}^k$ and $g_m : \Lambda_2^\infty \to \mathbb{N}^k$ for each $m \in \mathbb{N}^k$ satisfying

- $\sigma_{\Lambda_2}^{f_m(x)}(h(\sigma_{\Lambda_1}^m(x))) = \sigma_{\Lambda_2}^{f_m(x)+m}(h(x))$ for $x \in \Lambda_1^\infty$ and
- $\sigma_{\Lambda_1}^{g_m(y)}(h^{-1}(\sigma_{\Lambda_2}^m(y))) = \sigma_{\Lambda_1}^{g_m(y)+m}(h^{-1}(y))$ for $y \in \Lambda_2^\infty$.

Theorem (Carlsen–R)

Two k-graphs Λ_1 and Λ_2 are eventually one-sided conjugate if and only if there is an isomorphism $\Phi : \mathcal{G}_{\Lambda_1} \to \mathcal{G}_{\Lambda_2}$ satisfying $c_{\Lambda_2} \circ \Phi = c_{\Lambda_1}$.

Stabilised groupoids

Denote by $\mathcal{K}$ the compact operators on $\ell^2(\mathbb{N})$ generated by the rank-one operators $\{\theta_{i,j} : i, j \in \mathbb{N}\}$ and by $\mathcal{C}$ the diagonal subalgebra generated by $\{\theta_{i,i} : i \in \mathbb{N}\}$. For a commutative ring R with identity, we denote by $M_\infty(R)$ the ring of finitely supported, countable infinite square matrices over R, and by $D_\infty(R)$ the diagonal subalgebra consisting of diagonal matrices.

Denote by $\mathcal{N}$ the full countable equivalence relation $\mathcal{N} = \mathbb{N} \times \mathbb{N}$, regarded as a discrete principal groupoid with $(i,j)(j,k) = (i,k)$, $(i,j)^{-1} = (j,i)$ and unit space $\mathbb{N}$.

Given an ample groupoid $\mathcal{G}$, the product $\mathcal{G} \times \mathcal{N}$ is an ample groupoid under the product topology and coordinatewise operations. The unit space is identified with $\mathcal{G}^0 \times \mathbb{N}$. There are isomorphisms $C^*(\mathcal{G} \times \mathcal{N}) \cong C^*(G) \otimes \mathcal{K}$ such that $C_0(\mathcal{G}^0 \times \mathbb{N}) \cong C_0(\mathcal{G}^0) \otimes \mathcal{C}$ and $A_R(G \times \mathcal{N}) \cong A_R(G) \otimes M_\infty(R)$ such that $A_R(\mathcal{G}^0 \times \mathbb{N}) \cong A_R(\mathcal{G}^0) \otimes D_\infty(R)$.

For a k-graph Λ, there is a cocycle $\bar{c}_\Lambda : \mathcal{G}_\Lambda \times \mathcal{N} \to \mathbb{Z}^k$ given by $\bar{c}_\Lambda(\eta, (i,j)) = c_\Lambda(\eta)$ for $\eta \in \mathcal{G}_\Lambda$ and $(i,j) \in \mathcal{N}$.

Corollary (Carlsen–Ruiz–Sims–Tomforde, Carlsen–R)

Let Λ_1 and Λ_2 be k-graphs and let R be an integral domain with identity. TFAE

- *There is an isomorphism $\Phi : \mathcal{G}_{\Lambda_1} \times \mathcal{N} \to \mathcal{G}_{\Lambda_2} \times \mathcal{N}$ satisfying $\bar{c}_{\Lambda_2} \circ \Phi = \bar{c}_{\Lambda_1}$.*
- *There is a diagonal-preserving $*$-isomorphism $\Psi : C^*(\Lambda_1) \otimes \mathcal{K} \to C^*(\Lambda_2) \otimes \mathcal{K}$ satisfying $(\gamma_z^{\Lambda_2} \otimes \mathrm{Id}_\mathcal{K}) \circ \Psi = \Psi \circ (\gamma_z^{\Lambda_1} \otimes \mathrm{Id}_\mathcal{K})$ for $z \in \mathbb{T}^k$.*
- *There is a diagonal-preserving ring-isomorphism $\theta : \mathrm{KP}(\Lambda_1) \otimes M_\infty(R) \to \mathrm{KP}(\Lambda_2) \otimes M_\infty(R)$ such that $\theta(\mathrm{KP}(\Lambda_1)_n \otimes M_\infty(R)) = \mathrm{KP}(\Lambda_2)_n \otimes M_\infty(R)$ for $n \in \mathbb{Z}^k$.*

Let $\overline{\Omega}_k$ be the k-graph $\overline{\Omega}_k := \{(m, n) \in \mathbb{Z}^k \times \mathbb{Z}^k : m \leq n\}$ with degree map $d : \overline{\Omega}_k \to \mathbb{N}^k$ defined by $d(m, n) = n - m$, and range and source maps r, s defined by $r(m, n) = (m, m)$ and $s(m, n) = (n, n)$.

Let Λ be a row-finite k-graph with finitely many vertices and no sinks or sources. We write $\overline{X}_\Lambda$ for the space of all degree-preserving functors from $\overline{\Omega}_k$ to Λ. We equip $\overline{X}_\Lambda$ with the topology generated by subsets of the form

$$Z((m, n), \lambda) := \{x \in \overline{X}_\Lambda : x(m, n) = \lambda\}$$

where $(m, n) \in \overline{\Omega}_k$ and $\lambda \in \Lambda^{n-m}$.

Two-sided conjugacy of edge shifts

For $m \in \mathbb{Z}^k$, we denote by $\overline{\sigma}^m : \overline{X}_\Lambda \to \overline{X}_\Lambda$ the homeomorphism given by $\overline{\sigma}^m(x)(p,q) = x(p+m, q+m)$ for $x \in \overline{X}_\Lambda$ and $(p,q) \in \overline{\Omega}_k$.

Theorem (Carlsen–R)

Let Λ_1 and Λ_2 be row-finite k-graphs with finitely many vertices and no sinks or sources. TFAE

- *There is an isomorphism $\Phi : \mathcal{G}_{\Lambda_1} \times \mathcal{N} \to \mathcal{G}_{\Lambda_2} \times \mathcal{N}$ satisfying $\overline{c}_{\Lambda_2} \circ \Phi = \overline{c}_{\Lambda_1}$.*
- *There is a homeomorphism $h : \overline{X}_{\Lambda_1} \to \overline{X}_{\Lambda_2}$ such that $\overline{\sigma}^m \circ h = h \circ \overline{\sigma}^m$ for all $m \in \mathbb{N}^k$.*