Plan

LIMITS OF PROPER G-SPACES

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Statement of the theorem

Theorem

Let X be a measured G-space, where G is a measured groupoid. Then the following conditions are equivalent:

1. *X is an amenable G-space;*
2. *X is the limit of an equipped sequence of proper G-spaces.*

Projective limits of Banach spaces

Consider a sequence

$$B_0 \xleftarrow{P_1} B_1 \leftarrow \ldots B_{n-1} \xleftarrow{P_n} B_n \leftarrow \ldots$$

where B_n is a Banach space and P_n is a linear contraction.

Definition

Its projective limit is the Banach space

$$B_\infty = \varprojlim B_n = \{(b_n) : b_n \in B_n, b_{n-1} = P_n(b_n), \sup_n \|b_n\| < \infty\}$$

with the norm $\|(b_n)\| = \sup_n \|b_n\|$.

Sequences of spaces and AP equivalence relations

Given a sequence

$$X_0 \xrightarrow{\pi_{1,0}} X_1 \to \dots X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \dots$$

where X_n is a Borel space and $\pi_{n,n-1}$ a Borel surjection, we define the composition $\pi_n : X \to X_n$ and the equivalence relation

$$R_n = \{(x, y) \in X_0 \times X_0 : \pi_n(x) = \pi_n(y)\}$$

We obtain an increasing sequence of equivalence relations and define the equivalence relation $R = \cup R_n$.

Definition

- We say that R is the tail equivalence relation of the sequence.
- An AP equivalence relation is an equivalence relation obtained as a tail equivalence relation.

Limit of a sequence of spaces

Assume that X_0 is endowed with a measure class μ_0. Then X_n is endowed with the image class μ_n. One obtains the sequence:

$$L^\infty(X_0) \xleftarrow{(\pi_{1,0})^*} L^\infty(X_1) \leftarrow \ldots \leftarrow L^\infty(X_{n-1}) \xleftarrow{(\pi_{n,n-1})^*} L^\infty(X_n) \leftarrow \ldots$$

Identifying $L^\infty(X_n)$ to a subalgebra of $L^\infty(X_0)$, we have

$$\varprojlim L^\infty(X_n) = \bigcap_{n=0}^{\infty} L^\infty(X_n) = L^\infty(X_0)^R$$

which we may write as $L^\infty(X_\infty)$, where $X_\infty = X//R$ is also called the standard quotient. Loosely speaking, we say that:

X_∞ is the limit of the X_n's.

Equipped sequences and q-measures

We say that the above sequence of spaces (X_n) is equipped if the maps $\pi_{n,n-1}$ carry Borel systems of probability measures $q_{n,n-1}$. They define conditional expectations $q_n : B(X_0) \to B(X_n)$.

Definition

A measure μ_0 on X_0 is called a q-measure if it factors through q_n for all n: $\mu_0 = \mu_n \circ q_n$.

Two natural questions arise:

- problem 1: given $q = (q_{n,n-1})$, determine all q-measures;
- problem 2: characterize q-measures which are ergodic with respect to the above AP equivalence relation R.

Borel Bratteli diagrams

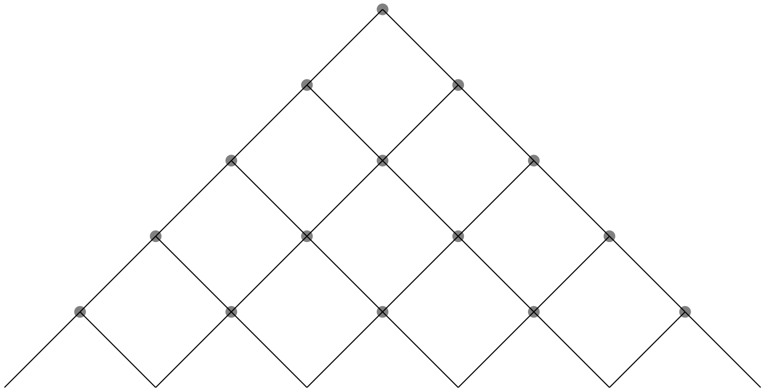
Some partial answers are known in the following setting.

Definition

- A Borel graph is a directed graph $r, s : E \rightrightarrows V$ where the sets of edges E and the set of vertices V are endowed with a Borel structure and the source and range maps are Borel.
- A Bratteli diagram is a directed graph $r, s : E \rightrightarrows V$ where $V = \coprod_{n=0}^{\infty} V(n)$, $E = \coprod_{n=1}^{\infty} E(n)$ and for each $n \geq 1$, $s(E(n)) = V(n-1)$ and $r(E(n)) = V(n)$.
- A Borel Bratteli diagram is a Bratteli diagram which is a Borel graph.

the Pascal triangle

Here is my favourite Bratteli diagram: the Pascal triangle.



Transition and cotransition probabilities

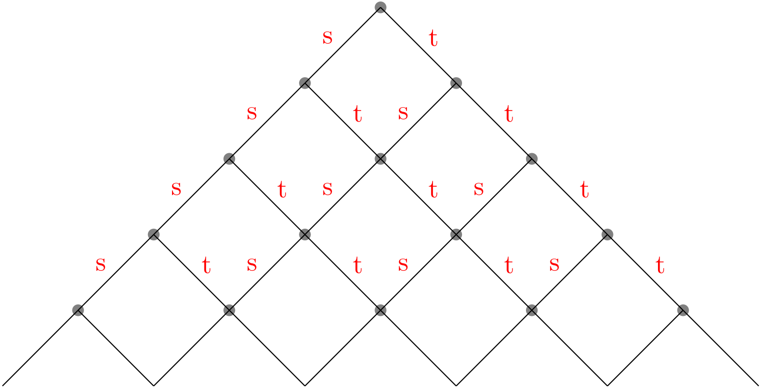
Definition

Let $E \rightrightarrows V$ be a Borel Bratteli diagram.

- A transition probability p assigns to each $v \in V(n-1)$ a probability measure p_v on $s^{-1}(v)$ and the map $v \mapsto p_v$ is Borel.
- A cotransition probability q assigns to each $w \in V(n)$ a probability measure q^w on $r^{-1}(w)$ and the map $w \mapsto q^w$ is Borel.

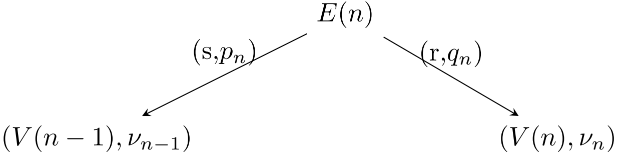
Transition probability on Pascal triangle

Let $0 < t < 1$ and $s = 1 - t$:



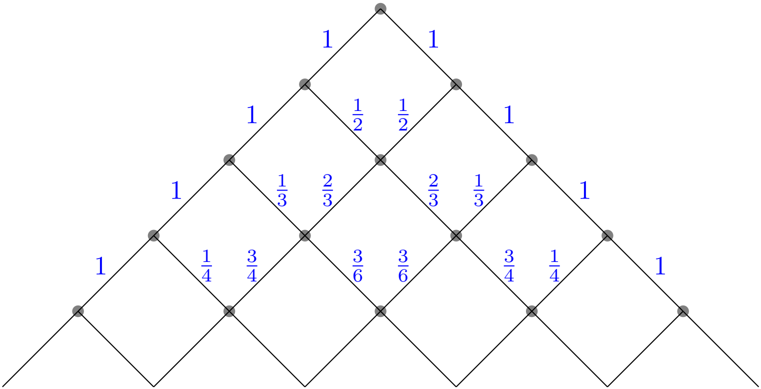
The cotransition probability of a random walk.

A random walk on a Bratteli diagram (V, E) is given by a transition probability p and an initial measure ν_0 on $V(0)$. One constructs by induction the measures ν_n on $V(n)$, called the one-dimensional distributions of the random walk. The cotransition probability of the random walk is given by the disintegration along the r-fibers of the lifted measure $\nu_{n-1} \circ p_n$ on $E(n)$:



$$\nu_{n-1} \circ p_n = \nu_n \circ q_n$$

Cotransition probability on the Pascal triangle



Note that the cotransition probability does not depend on t.

The path spaces of a Bratteli diagram

Let (V, E) be a Borel Bratteli diagram equipped with a Borel cotransition probability $q = (q^w), w \in V$.
Define
$X_0 = E(1) * E(2) * \ldots = \{e_1 e_2 \ldots, \text{infinite path of the diagram}\}$
$X_n = E(n+1) * E(n+2) * \ldots = \{e_{n+1} e_{n+2} \ldots\}$ and
$\pi_{n,n-1}(e_n e_{n+1} \ldots) = e_{n+1} e_{n+2} \ldots$.
The cotransition probability q defines expectations
$q_n : B(X_0) \to B(X_n)$.
In the above example of the Pascal triangle, the q-measures are the invariant measures. The ergodic probability measures are the Markov measures μ_t defined by the transition probability $t \in (0, 1)$.

Bounded harmonic sequences

Let (V, E) be a Borel Bratteli diagram equipped with a Borel transition probability $p = (p_v), v \in V$ and an initial measure class ν_0 on $V(0)$. It propagates to measure classes ν_n on $V(n)$. The elements of the projective limit of the sequence

$$L^\infty(V(0)) \xleftarrow{p_1} L^\infty(V(1)) \leftarrow \ldots \leftarrow L^\infty(V(n-1)) \xleftarrow{p_n} L^\infty(V(n)) \leftarrow \ldots$$

where p_n is the Markov operator defined for $h \in L^\infty(V(n))$ and $v \in V(n-1)$ by

$$p_n(h)(v) = \int (h \circ r_n) dp_v$$

are the bounded harmonic sequences.

The Poisson boundary

Definition

The Poisson boundary of the random walk defined by (p, ν_0) is the projective limit $H(p) = \varprojlim L^\infty(V(n))$.

The Poisson boundary of the random walk has another interpretation related to the above problem 2. Introduce the infinite path spaces $X_0 = E(1) * E(2) * \ldots$, $X_n = E(n+1) * E(n+2) * \ldots$ and $\pi_{n,n-1} : X_{n-1} \to X_n$. Endow X_0 [resp. X_n] with the class of the Markov measure μ_0 [resp. μ_n] defined by $(p, \nu_0))$ [resp. $(p_{|n}, \nu_n))$] and define $L^\infty(X_\infty) = \bigcap_n L^\infty(X_n)$ as above.

The Neveu theorem

Theorem (Neveu 1964)

The ordered Banach spaces $H(p)$ and $L^\infty(X_\infty)$ are canonically isomorphic.

The commutative diagram:

$$\begin{array}{ccc} L^\infty(X_{n-1}) & \xleftarrow{(\pi_{n,n-1})^*} & L^\infty(X_n) \\ P_{n-1}\downarrow & & P_n\downarrow \\ L^\infty(V(n-1)) & \xleftarrow{p_n} & L^\infty(V(n)) \end{array}$$

where $P_n^v = p_n^v p_{n+1} \ldots$ defines $P_\infty : L^\infty(X_\infty) \to H(p)$.

Proof of the Neveu theorem

The inverse map can be constructed using cotransition probabilities as follows:
We fix an initial measure $\underline{\nu}_0$ on $V(0)$. This gives the measures $\underline{\nu}_n$ on $V(n)$ and the Markov measures $\underline{\mu}_n$ on X_n. Then $f = f_n \circ \pi_n \in L^\infty(X_\infty)$ and $h = (h_n) \in H(p)$ are related by

$$h_n \underline{\nu}_n = s_{n*}(f_n \underline{\mu}_n) \quad \text{with} \quad s_n : X_n \to V(n)$$

Given a bounded harmonic sequence $h = (h_n) \in H(p)$, consider the n-th distributions $\nu_n = h_n \underline{\nu}_n$ and use the cotransition probabilities to construct the corresponding Markov measure μ_0 on X_0. Then f is defined by $\mu_0 = f \underline{\mu}_0$.

Amenable and proper groupoids

Definition

A measured groupoid (G, λ, μ) is called:

1. amenable if it has an invariant mean, i.e. a unital positive linear map $p : L^\infty(G) \to L^\infty(G^{(0)})$ such that for all $\varphi \in L^I(G)$, $h \in L^\infty(G^{(0)})$, $f \in L^\infty(G)$, $p((h \circ r)f) = hp(f) \quad \text{and} \quad p(\varphi * f) = \varphi * p(f)$.
2. proper if it has an invariant mean which is continuous with respect to the weak$*$-topologies.

A continuous invariant mean is of the form $p(f)(x) = \int f dp^x$ where (p^x) is a Haar system consisting of probability measures.

Amenable G-spaces

Definition

We say that a measured G-space X is amenable [resp. proper] if the semi-direct product $G \ltimes X$ is amenable [resp. proper].

This is Zimmer's notion of amenability. For example, [Zimmer,1978] showed that the Poisson boundary of a random walk on a group G is an amenable G-space. It should not be confused with Greenleaf's definition of an amenable G-space, which is the existence of an invariant mean $L^\infty(X) \to L^\infty(G^{(0)})$.

Amenable actions on von Neuman algebras

By definition, a G-von Neumann algebra is a von Neumann algebra of the form $L^\infty(X, \mathcal{M})$ where $\mathcal{M}$ is a measurable G-bundle of von Neumann algebras over a measured G-space X.

C. Anantharaman-Delaroche introduced in 1979 the following definition in the case of a locally compact group G:

Definition

A G-bundle $\mathcal{M}$ of von Neumann algebras over $G^{(0)}$ is called amenable if there exists an invariant mean $m : L^\infty(G, r^*\mathcal{M}) \to L^\infty(G^{(0)}, \mathcal{M})$.

Amenable actions on von Neuman algebras, cont'd

and proved the group case of

Proposition

Let $\mathcal{N}$ be a G-subbundle of a G-bundle $\mathcal{M}$. If

1. *the G-bundle $\mathcal{M}$ is amenable and*
2. *there exists a G-expectation $E : L^\infty(G^{(0)}, \mathcal{M}) \to L^\infty(G^{(0)}, \mathcal{N})$,*

then, the G-bundle $\mathcal{N}$ is amenable.

Approximate invariant mean

There is an equivalent definition of an amenable measured groupoid in terms of approximate invariant mean.

Definition

An approximate invariant mean for a measured groupoid (G, λ, μ) is a sequence (p_n) where for each n, $p_n = (p_n^x)$ is a Borel system of probability measures on the fibers of $r : G \to G^{(0)}$ with $p_n^x \prec \lambda^x$ a.e. and $\|\gamma p_n^{r(\gamma)} - p_n^{s(\gamma)}\|_1$ tends to 0 a.e.

The existence of an approximate invariant mean is called Reiter's property. Proper amenability is the case of exact invariance.

Proposition

A measured groupoid (G, λ, μ) is amenable if and only if it possesses an approximate invariant mean.

The Chu-Li theorem

It was shown for amenable groups ([Kaimanovich-Vershik, 1983] and [Rosenblatt, 1981]) and more recently for amenable groupoids that one can find an approximate invariant mean of a special form.

Theorem (Chu-Li, 2014)

Let (G, λ, μ) be a measured groupoid. Then the following conditions are equivalent:

1. (G, λ, μ) *is amenable;*
2. *there exists an approximate invariant mean of the form* $p_n = p * p * \ldots * p$ *(n-th convolution product) where* $p = (p^x)$ *is a Borel system of probability measures on the fibers of* r *with* $p_n^x \prec \lambda^x$ *a.e.*

Comments and examples

The convolution of two r-systems of measures p, q (also called transverse functions) is defined in [Connes, 1978] by

$$(p * q)^x = \int (\gamma q^{s(\gamma)}) dp^x(\gamma)$$

1) When $G = X \times X$, where X is a finite or countable set, an r-system of probability measures $p = (p^x)$ is exactly a stochastic matrix P. Then (p_n) is an approximate invariant mean iff $P^n(x, y) - P^n(x, z)$ goes to zero for all x, y, z. This happens iff 1 is a simple eigenvalue of P and the only eigenvalue of modulus 1.

2) When G is an abelian locally compact group, every probability measure $p \prec \lambda$ whose support generates G gives an approximate invariant mean.

$2 \Rightarrow 1$

Consider an equipped sequence of G-spaces

$$X_0 \xrightarrow{\pi_{1,0}} X_1 \to \ldots X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \ldots$$

where we assume that the maps $\pi_{n,n-1}$ and the systems of probability $q_{n,n-1}$ are are G-equivariant.

Then the von Neumann algebras $M_n = L^\infty(X_n)$ and $M_\infty = L^\infty(X_\infty)$ are G-algebras. We have compatible expectation $q_n : M_0 \to M_n$. Limit points exist and are expectations $q_\infty : M_0 \to M_\infty$. By assumption, the action of G on M_0 is amenable. By the result of [Anantharaman-Delaroche], so is the action of G on $L^\infty(X_\infty)$.

$1 \Rightarrow 2$

Assume that (X, μ) is a measured amenable G-space, where G is a Borel groupoid with Haar system. Thus the groupoid $G \ltimes X$ is amenable. To simplify the notation, we may as well assume that $X = G^{(0)}$ and that G is an amenable groupoid.

By the result of [Chu-Li], there exists $p = (p^x)$ such that (p_n) is an approximate invariant mean. Form the stationary Bratteli diagram (V, E), where $V(n) = G$ and $E(n) = \{g, \gamma, g\gamma) : (g, \gamma) \in G^{(2)}\}$. Equip it with the transition probability p and the initial measure μ. It is easy to show that p has the Liouville property, i.e. that $H(p)$ is reduced to $L^\infty(G^{(0)})$.

$1 \Rightarrow 2$ cont'd

There is a left action of G on (V, E) leaving p invariant. Therefore our sequence (X_n) of infinite path spaces is G-equivariant. Moreover, the G-spaces X_n are proper (the action is essentially the left translation of G on itself). This exhibits $G^{(0)} = X_\infty$ as a limit of proper G-spaces. Moreover, the sequence can be equipped with G-invariant systems of probability measures.

Mackey range

The Bratteli diagram (V, E), where $V(n) = G$ and $E(n) = \{g, \gamma, g\gamma) : (g, \gamma) \in G^{(2)}\}$ is the skew-product graph obtained from the Bratteli diagram (W, F), where $W(n) = G^{(0)}$, $F(n) = G$ and the tautological labeling $F(n) \to G$. Therefore, the tail equivalence relation R on the infinite path space X_0 of (V, E) is the skew-product $S(c)$ of the tail equivalence relation S on the infinite path space Y_0 of (W, F) by the cocycle $c : S \to G$ defined by the labeling. By definition, the Mackey range of c is the standard quotient $X_0//R = X_\infty$. Thus,

Corollary (Elliott-Giordano, 1993; Adams-E-G, 1994)

Let Γ be a locally compact group. Every amenable Γ-space X is the Mackey range of a cocycle $c : S \to \Gamma$, where S is an AP equivalence relation.

Some references

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The End

Thank you for your attention!