Purely infinite simple groupoid algebras

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Infinite projections vs infinite idempotents

ANALYSIS	ALGEBRA
Let A be a *simple* C^*-algebra	Let R be a *algebraically simple* ring
Partial order $\leq$ on A^+ where $a \leq b \iff b - a \in A^+$	Partial order $\leq_a$ on $R \setminus \{0\}$ where $a \leq_a b \iff ab = ba = a$
Let $p \in A$ be a projection. $p = p^2 = p^*p$	Let $e \in R$ be an idempotent. $e^2 = e$
p is an infinite projection $\iff$ there exists $x \in A$ such that $xx^* < x^*x = p$	e is an infinite idempotent $\iff$ there exist $x, y \in R$ such that $yx <_a xy = e$
A is purely infinite simple if every hereditary subalgebra contains an infinite projection	R is algebraically purely infinite simple if every right ideal contains an infinite idempotent

Ample groupoids

- Let G be an ample Hausdorff groupoid.
  - That means G is Hausdorff and has a basis of compact open bisections.
  - Equivalently, G is Hausdorff, étale and $G^{(0)}$ is totally disconnected.
- Examples:
  - a discrete group;
  - the groupoid associated to a directed graph;
  - the groupoid associated to a higher rank graph;
  - the groupoid associated to a category of paths (Spielberg);
  - the groupoid associated to a group acting on a graph ** (Exel-Pardo);
  - the groupoid associated to a LCSC (Spielberg)**;
  - the tight groupoid of an inverse semigroup representation (Exel)**.

  ** Not necessarily Hausdorff.

Analysis: Groupoid C^*-algebras

Let G be an ample Hausdorff groupoid.

How to build $C^*(G)$:

- Start with the vector space $C_c(G)$.
- Define a convolution and involution in a way that incorporates the operations in G.
- Define a norm on $C_c(G)$ that satisfies the C^*-identity.
- Complete.

Convolution and Involution

- $C_c(G) := \{f : G \to \mathbb{C} \mid f \text{ is continuous and has compact support }\}$.
- Then $C_c(G)$ is a complex vector space where addition and scalar multiplication are defined pointwise.
- Convolution:
$$f * g(\gamma) = \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma)$$
for $f, g \in C_c(G)$ and $\gamma \in G$.
- Involution:
$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$
for $f \in C_c(G)$ and $\gamma \in G$.

A norm

- For $f \in C_c(G)$ the universal norm of f is

$$\|f\| := \sup\{\|\pi(f)\|_{op} \mid \pi \text{ is a representation of } C_c(G)\}$$

where a representation of $C_c(G)$ is a $*$-homomorphism from $C_c(G)$ to $B(\mathcal{H})$ for some $\mathcal{H}$ that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $B(\mathcal{H})$.

Theorem (Renault, Muhly-Williams)

The function $\|\cdot\|$ defines a C^*-norm on $C_c(G)$

- The completion of $C_c(G)$ with respect to this norm is a C^*-algebra, denoted $C^*(G)$.
- From now on we will assume G is second countable and that $C^*(G) \cong C^*_r(G)$.

P.I.S. C^*-algebras are always groupoid algebras

Theorem (Spielberg, Exel-Pardo, Katsura, Kirchberg-Phillips)

Suppose A is a purely infinite simple C^*-algebra in UCT. Then there exists an ample Hausdorff groupoid G such that A is stably isomorphic to $C^*(G)$. If A is unital, then $A \cong C^*(G)$.

(Note: there is an error in the above theorem.)

Open question: Can we find necessary and sufficient conditions on G to ensure that $C^*(G)$ is purely infinite simple?

Some nice elements of $C^*(G)$

- $C_c(G) \subseteq C^*(G)$
- $C_c(G^{(0)}) \subseteq C^*(G)$
    - Here convolution is just point-wise multiplication
- $1_U \in C^*(G)$ where $U \subseteq G^{(0)}$ is a compact open
    - Let $U, V \subseteq G^{(0)}$, then $1_U 1_V = 1_{U\cap V}$
    - $(1_U)^* = 1_U$

Theorem (J. Brown-C-Sierakowski)

Suppose G is an ample Hausdorff groupoid that is minimal and effective (such that ...). Then $C^*(G)$ is purely infinite simple if and only if 1_U is an infinite projection for every compact open $U \subseteq G^{(0)}$.

Open question: For compact open $U \subseteq G^{(0)}$, can we find a groupoid condition that ensures 1_U is an infinite projection?

Infinite projections vs infinite idempotents

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A sufficient condition

- For now, suppose $C^*(G)$ is simple.
- Fix compact open $U \subseteq G^{(0)}$.
- We call a compact open bisection $B_U \subseteq G$ a contracting bisection for U if $r(B_U) \subsetneq s(B_U) \subseteq U$.
- If there exists a contracting bisection B_U for U, then 1_U is an infinite projection: $1_{B_U}(1_{B_U})^* = 1_{r(B_U)}, \quad (1_{B_U})^* 1_{B_U} = 1_{s(B_U)}$ and

$$1_{r(B_U)} < 1_{s(B_U)} \leq 1_U.$$

- We say G is locally contracting if there exists a contracting bisection B_U for every compact open $U \subseteq G^{(0)}$.
- Corollary to BCS Theorem: If G is locally contracting, then $C^*(G)$ is purely infinite simple.
- Open question: Does the converse hold?

A new approach

- Can we use purely algebraic information to gain insight into the C^*-algebra?

- We want a purely algebraic groupoid algebra $A(G)$ that is defined without a norm such that C^*-algebraic properties of $C^*(G)$ correspond with the analogous algebraic properties of $A(G)$.

Algebra: Steinberg algebras

- The complex Steinberg algebra $A(G)$ is the $*$-subalgebra of $C_c(G)$ such that

$$A(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}$$

- Convolution: $1_B 1_D = 1_{BD}$
- Involution: $(1_B)^* = 1_{B^{-1}}$
- $A(G)$ is dense in $C^*(G)$
- The class of Steinberg algebras includes Leavitt path algebras and Kumjian-Pask algebras as special cases

Theorem (J.Brown-C-Farthing-Sims)

Suppose G is an ample Hausdorff groupoid (such that ...).
The following are equivalent:

1. *G is minimal and effective;*
2. *$A(G)$ is (algebraically) simple; and*
3. *$C^*(G)$ is simple.*

More open questions:

- Is there a similar kind of theorem for purely infinite simple algebras?
- Can we find necessary and sufficient conditions on G to ensure that $A(G)$ is algebraically purely infinite simple?

Some progress

Theorem (J.Brown-C-an Huef)

Suppose G is an ample Hausdorff groupoid that is minimal and effective. Then $A(G)$ is algebraically purely infinite simple if and only if 1_U is an infinite idempotent for every compact open $U \subseteq G^{(0)}$.

Open question: What is the relationship between 1_U being an infinite idempotent in $A(G)$ and 1_U being an infinite projection in $C^*(G)$?

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Locally contracting revisited

Corollary

Suppose G is an ample Hausdorff groupoid that is minimal and effective. If G is locally contracting, then $A(G)$ is algebraically purely infinite simple.

The converse is not known.

- Locally contracting is a purely algebraic condition: If G is locally contracting, then the C^*-infiniteness of a projection 1_U is achieved using an element of the Steinberg algebra $x = 1_{B_U}$.
- More open questions:
  - Can 1_U be an infinite idempotent in $A(G)$ where the 'witnesses' are not of the form x and x^*?
  - Can 1_U be an infinite projection in $C^*(G)$ where the 'witness' x is not an element of the Steinberg algebra?

Main theorem

Theorem (J.Brown-C-an Huef)

Suppose G is an ample Hausdorff groupoid that is minimal and effective. If $A(G)$ is algebraically purely infinite simple, then $C^*(G)$ is purely infinite simple.

The converse is not known.

Corollary

If every 1_U is an infinite idempotent in $A(G)$, then every 1_U is an infinite projection in $C^*(G)$.

The converse is not known.

Sketch of Proof of Main theorem

- Simplicity follows from earlier results.
- Define
  $B(G) := \{1_U x 1_U : x \in C^*(G) \text{ and } U \subseteq G^{(0)} \text{ compact open set}\}$
- Then $B(G)$ is an s-unital dense subalgebra of $C^*(G)$.
- If $A(G)$ is algebraically purely infinite simple, then $B(G)$ is algebraically 'properly infinite'.
- Since $C^*(G)$ is simple, the ideal generated by x is all of $C^*(G)$ for any $x \in C^*(G)$.
- For any $x \in C^*(G)$, we have $B(G)xB(G) \subseteq B(G)$ and $\overline{B(G)xB(G)} = C^*(G)$.
- We use this to show that all positive elements of $C^*(G)$ are 'infinite' using an approximation argument.

The End

Thank you!