On Fell bundles over inverse semigroups and their left regular representations

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Based on joint work with E. Bédos [1]

Fell bundles over inverse semigroups

Fell bundles $\mathcal{A} = \{A_s\}_{s \in S}$ over inverse semigroups S were defined by Exel [4] following unpublished work by Sieben.

$C^*(\mathcal{A})$ was defined as the universal C^*-algebra for C^*-representations of $\mathcal{A}$.

Buss, Exel [2]: To each twisted partial action β of an inverse semigroup S on a C^*-algebra A, one may associate a Fell bundle $\mathcal{A}_\beta$ over S such that

$$C^*(\mathcal{A}_\beta) \simeq A \rtimes_\beta S$$

Buss, Exel [3]: Any Fell bundle $\mathcal{B}$ over an étale groupoid $\mathcal{G}$ gives rise to a Fell bundle $\mathcal{A}$ over S, where S is any inverse semigroup consisting of bisections (or slices) of $\mathcal{G}$ (as defined by Renault in [6]), and under some mild assumptions

$$C^*(\mathcal{B}) \simeq C^*(\mathcal{A})$$

Fell bundles over inverse semigroups

Throughout this talk, consider a fixed inverse semigroup S with idempotent semilattice $E \subset S$.

A Fell bundle over S is a quadruple

$$\mathcal{A} = \left(\{A_s\}_{s\in S}, \{\mu_{s,t}\}_{s,t\in S}, \{*_s\}_{s\in S}, \{j_{t,s}\}_{s,t\in S, s\leq t}\right)$$

where for $s, t \in S$ we have that

(i) A_s is a complex Banach space,
(ii) $\mu_{s,t} : A_s \odot A_t \to A_{st}$ is a linear map,
(iii) $*_s : A_s \to A_{s^*}$ is a conjugate linear isometric map,
(iv) $j_{t,s} : A_s \hookrightarrow A_t$ is a linear isometric map whenever $s \leq t$.

Fell bundles over inverse semigroups

It is moreover required that for every $r, s, t \in S$, and every $a \in A_r$, $b \in A_s$, and $c \in A_t$, we have

(v) $\mu_{rs,t}(\mu_{r,s}(a \otimes b) \otimes c) = \mu_{r,st}(a \otimes \mu_{s,t}(b \otimes c))$,
(vi) $*_{rs}(\mu_{r,s}(a \otimes b)) = \mu_{s^*,r^*}(*_s(b) \otimes *_r(a))$,
(vii) $*_{s^*}(*_s(a)) = a$,
(viii) $\|\mu_{r,s}(a \otimes b)\| \leq \|a\| \, \|b\|$,
(ix) $\|\mu_{r^*,r}(*_r(a) \otimes a)\| = \|a\|^2$,
(x) $\mu_{r^*,r}(*_r(a) \otimes a) \geq 0$ in A_{r^*r},
(xi) if $r \leq s \leq t$, then $j_{t,r} = j_{t,s} \circ j_{s,r}$,
(xii) if $r \leq r'$ and $s \leq s'$, then $j_{r's',rs} \circ \mu_{r,s} = \mu_{r',s'} \circ (j_{r',r} \otimes j_{s',s})$ and $j_{s',s} \circ *_s = *_{s'} \circ j_{s',s}$.

Exel: axioms (i)-(ix) imply that A_e is a C^*-algebra whenever $e \in E$. So axiom (x) is meaningful. Moreover the following properties hold:

(xiii) $j_{s,s}$ is the identity map id_{A_s} for every $s \in S$;
(xiv) If $e, f \in E$ and $e \leq f$, then $j_{f,e}(A_e)$ is an ideal in A_f.

Pre-representations and representations

A *pre-representation* of $\mathcal{A} = (\{A_s\}_{s\in S}$ in a complex $*$-algebra B is a family $\Pi = \{\pi_s\}_{s\in S}$, where for each $s \in S$,

$$\pi_s : A_s \to B$$

is a linear map such that for all $s, t \in S$, all $a \in A_s$, and all $b \in A_t$, one has

(a) $\pi_{st}(a \cdot b) = \pi_s(a)\pi_t(b)$,
(b) $\pi_{s^*}(a^*) = \pi_s(a)^*$.

If in addition Π satisfies

(c) $\pi_t \circ j_{t,s} = \pi_s$ whenever $s \leq t$,

then Π is called a *representation* of $\mathcal{A}$ in B.

The algebra of finitely supported sections

Consider the direct sum of vector spaces

$$\mathcal{C}_c(\mathcal{A}) = \bigoplus_{s \in S} A_s$$

Write $g \in \mathcal{C}_c(\mathcal{A})$ as a formal sum $g = \sum_{s\in S} a_s\delta_s$ where $a_s \in A_s$ for $s \in S$ and $a_s = 0$ for all but finitely many s. Give $\mathcal{C}_c(\mathcal{A})$ the structure of a complex $*$-algebra by extending linearly

$$(a_s\delta_s)(b_t\delta_t) = (a_s \cdot b_t)\delta_{st}$$
$$(a_s\delta_s)^* = a_s^*\delta_{s^*}$$

To each pre-representation Π of $\mathcal{A}$ in a $*$-algebra B one may associate a $*$-homomorphism $\Phi_\Pi : \mathcal{C}_c(\mathcal{A}) \to B$ given by

$$\Phi_\Pi\Big(\sum_{s\in S} a_s\delta_s\Big) = \sum_{s\in S} \pi_s(a_s)\,,$$

The map $\Pi \mapsto \Phi_\Pi$ gives a bijection between pre-representations of $\mathcal{A}$ in B and $*$-homomophisms from $\mathcal{C}_c(\mathcal{A})$ into B.

Full cross-sectional C^*-algebras of Fell bundles over inverse semigroups

Exel defines $C^*(\mathcal{A})$ as the universal C^*-algebra for representations of $\mathcal{A}$ in C^*-algebras.

Let $\mathcal{N}_\mathcal{A}$ be the span of elements of the form

$$a_t\delta_t - j_{s,t}(a_t)\delta_s \in \mathcal{C}_c(\mathcal{A})$$

with $s, t \in S$, $t \leq s$ and $a_t \in A_t$. Exel [4]: $\mathcal{N}_\mathcal{A}$ is a *-ideal in $\mathcal{C}_c(\mathcal{A})$. Moreover, every $*$-homomorphism from $\mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A}$ to a C^*-algebra B corresponds to a $*$-homomorphism from $C^*(\mathcal{A})$ to B and vice versa.

Let $C^*_{\mathrm{KS}}(\mathcal{A})$ be the universal C^*-algebra for *-representations of $\mathcal{C}_c(\mathcal{A})$ in C^*-algebras. Let $\mathcal{M}_\mathcal{A}$ be the closure of $\mathcal{N}_\mathcal{A}$ in $C^*_{\mathrm{KS}}(\mathcal{A})$. Bédos, N. [1]: $C^*_{\mathrm{KS}}(\mathcal{A})/\mathcal{M}_\mathcal{A}$ is naturally isomorphic to $C^*(\mathcal{A})$.

Exel's reduced crossed-sectional C^*-algebra of a Fell bundle over an inverse semigroup

Sketch of the construction of $C_r^*(\mathcal{A})$ following [4]: Let $\mathcal{E}$ be the restriction of $\mathcal{A}$ to the canonical semilattice $E \subset S$.

Every pure state φ on $C^*(\mathcal{E})$ can be restricted to A_e for each $e \in E$, and extended again to A_s for each $s \in S$ in a way such that one can define a state $\tilde{\varphi}$ on $\mathcal{C}_c(\mathcal{A})$. Moreover, $\|\tilde{\varphi}\| \leq \|\varphi\|$ and $\tilde{\varphi}$ vanishes on the ideal $\mathcal{N}_{\mathcal{A}}$.

One can then form the GNS-representation $\Upsilon_{\tilde{\varphi}}$ of $\tilde{\varphi}$. $C_r^*(\mathcal{A})$ is defined as the Hausdorff completion of $\mathcal{C}_c(\mathcal{A})$ w.r.t. the C^*-seminorm

$$\|g\|_r' = \sup_{\varphi} \|\Upsilon_{\tilde{\varphi}}(g)\|$$

with the supremum taken over all pure states on $C^*(\mathcal{E})$.

The left regular representation

For $e \in E$, let $S_e = \{s \in S : s^*s = e\}$. Define the right Hilbert A_e-module

$$X_e = \left\{ \xi \in \prod_{s \in S_e} A_s : \sum_{s \in S_e} \xi(s)^* \cdot \xi(s) \text{ is norm convergent in } A_e \right\}$$

For any $t \in S$, let $\lambda_{e,t} : A_t \to \mathcal{L}(X_e)$ be given by

$$(\lambda_{e,t}(a_t)\xi)(v) = \begin{cases} a_t \cdot \xi(t^*v) & \text{if } vv^* \leq tt^* \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Lambda^e = \{\lambda_{e,t}\}_{t \in S}$ is a pre-representation of $\mathcal{A}$ in $\mathcal{L}(X_e)$, and we call $\Lambda := \prod_{e \in E} \Lambda^e$ the left regular pre-representation of $\mathcal{A}$. This gives rise to a $*$-homomorphism

$$\Phi_\Lambda : \mathcal{C}_c(\mathcal{A}) \to \prod_{e \in E} \mathcal{L}(X_e)$$

satisfying $\Phi_\Lambda \left(\sum_{t \in S} a_t \delta_t \right) = \left(\sum_{t \in S} \lambda_{e,t}(a_t) \right)_{e \in E}$

The left regular representation

Theorem 1 (Bédos, N., extending Wordingham's Theorem [5])

The left regular representation Φ_Λ of $\mathcal{C}_c(\mathcal{A})$ is injective.

We define $C^*_{r,\mathrm{KS}}(\mathcal{A})$ as the C^*-algebra generated by the image of Φ_Λ. Let $\mathcal{I}_\mathcal{A}$ be the closure of $\Phi_\Lambda(\mathcal{N}_\mathcal{A})$.

Question 1

Is there some relationship between $C^*_r(\mathcal{A})$ and $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{I}_\mathcal{A}$?

Fell bundles over semilattices

Let $\mathcal{E}$ be a Fell bundle over a semilattice.

Proposition 2 (Bédos, N.)

We have $C^*_{r,\mathrm{KS}}(\mathcal{E}) = C^*_{\mathrm{KS}}(\mathcal{E})$.

Corollary 3

$C^*_r(\mathcal{E}) = C^*_{r,\mathrm{KS}}(\mathcal{E})/\mathcal{I}_{\mathcal{E}}$

The conditional expectation

More generally, let $\mathcal{A}$ be a Fell bundle over S, and let $\mathcal{E}$ be the restriction of $\mathcal{A}$ to the canonical semilattice $E \subset S$.

For any $s \in S$, let $\gamma_s : A_s \to X_{s^*s}$ be the natural inclusion. Then γ_s is an isometry and a right A_e Hilbert module map. Define $\mathfrak{E}_{\mathrm{KS}} : C^*_{r,\mathrm{KS}}(\mathcal{A}) \to \prod_{e \in E} \mathcal{L}(A_e)$ by

$$\mathfrak{E}_{\mathrm{KS}}((T_e)_{e \in E}) = (\gamma_e^* T_e \gamma_e)_{e \in E}$$

Proposition 4 (Bédos, N.)

The map $\mathfrak{E}_{\mathrm{KS}}$ is faithful. If S is E^*-unitary, then $\mathfrak{E}_{\mathrm{KS}}$ is a faithful conditional expectation $C^*_{r,\mathrm{KS}}(\mathcal{A}) \to C^*_{r,\mathrm{KS}}(\mathcal{E})$.

Comparison with Exel's reduced C^*-algebra

Suppose from now on that S is E^*-unitary. Let

$$\mathcal{J}_{\mathcal{A}} = \{T \in C^*_{r,\mathrm{KS}}(\mathcal{A}) : \mathfrak{E}_{\mathrm{KS}}(T^*T) \in \mathcal{I}_{\mathcal{E}}\}.$$

Theorem 5 (Bédos, N.)

We have

(i) $\mathcal{J}_{\mathcal{A}}$ *is a closed two-sided ideal of* $C^*_{r,\mathrm{KS}}(\mathcal{A})$ *with* $\mathcal{I}_{\mathcal{A}} \subset \mathcal{J}_{\mathcal{A}}$.

(ii) $C^*_r(\mathcal{A})$ *is naturally isomorphic to* $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}}$.

Corollary 6

$C^*_r(\mathcal{A})$ *is naturally isomorphic to* $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{I}_{\mathcal{A}}$ *if and only if* $\mathcal{I}_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}$.

Example 7

If there exists an idempotent pure grading $\phi : S^\times \to G$ where G is an exact group, then $\mathcal{I}_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}$, so $C^*_r(\mathcal{A}) \simeq C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{I}_{\mathcal{A}}$

References I

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Thank you for listening!