Equilibrium states on right LCM semigroup C*-algebras revisited

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joint with N. Brownlowe, J. Ramagge and N. Stammeier

The KMS condition for finite systems

Finite quantum systems: a time evolution on $M_n(\mathbb{C})$ is given by a one-parameter group of automorphisms

$$\sigma_t(a) = e^{itH} a e^{-itH},$$

where $t \in \mathbb{R}$, $a \in M_n(\mathbb{C})$ and H is a self-adjoint matrix.

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The *Gibbs state* at $\beta > 0$ is $\varphi_G(a) = \frac{\mathrm{Tr}(ae^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$. It satisfies

$$\varphi_G(ab) = \varphi_G(b\sigma_{i\beta}(a)), \tag{1}$$

for $a, b \in M_n(\mathbb{C})$ *analytic*, i.e. $t \mapsto \sigma_t(a)$ extends to an entire function on $\mathbb{C}$.

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(1) - the *KMS condition*, cf. *Haag-Hugenholtz-Winnick* (1967): equilibrium for a state on a C^*-algebra with time evolution.

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$\mathcal{A}$ C^*-algebra, $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ time evolution, φ a state on A.

1. φ is KMS$_\beta$ (at inverse temperature $\beta \in [0, \infty)$) if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

   for all $a, b \in \mathcal{A}^a$, the dense $*$-subalgebra of *analytic* elements.

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2. A state φ is a *ground state* if for all a, b with b analytic, the function $z \to \varphi(a\sigma_z(b))$ is bounded in the upper-half plane.

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3. KMS_∞ if $\varphi = w^* \lim \varphi_n$ as $\beta_n \to \infty$ and φ_n is KMS_{β_n}.

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References: Bratteli-Robinson, Pedersen, Connes-Marcolli.

KMS$_\infty$ strictly subset of ground states

Theorem (Laca-Raeburn (2010))

$C^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ is the universal C^*-algebra generated by isometries s and $\{v_p \mid p \text{ prime}\}$, subject to the relations

1. $v_p s = s^p v_p$;
2. $v_p v_q = v_q v_p$,
3. $v_p^* v_q = v_q v_p^*$ *when* $p \neq q$,
4. $s^* v_p = s^{p-1} v_p s^*$, *and*
5. $v_p^* s^k v_p = 0$ *for* $1 \leq k < p$.

Dynamics: $\sigma_t(s) = s$ *and* $\sigma_t(v_p) = p^{it} v_p$.

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Dynamics: $\sigma_t(s) = s$ *and* $\sigma_t(v_p) = p^{it} v_p$*. Then, for* $\beta < 1$*, there are no KMS states, if* $\beta \in [1, 2]$*, there is a unique* KMS_β *state;*

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Dynamics: $\sigma_t(s) = s$ and $\sigma_t(v_p) = p^{it} v_p$. Then, for $\beta < 1$, there are no KMS states, if $\beta \in [1, 2]$, there is a unique KMS_β state; if $\beta \in (2, \infty]$, the KMS_β states are parametrised by probability measures on $\mathbb{T}$ while the ground states are parametrised by states on the Toeplitz C^*-algebra generated by *a single isometry.*

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Theorem (Afsar-Brownlowe-L-Stammeier (2016))

Let S be a right LCM monoid and $N : S \to \mathbb{N}^\times$ homomorphism such that S is admissible. Consider the time evolution $\sigma_t(v_s) = N_s^{it} v_s$. If $\beta_c \in \mathbb{R}$ is such that the function

$$\zeta_N(\beta) := \sum_{n \in Irr(N(S))} n^{-(\beta-1)},$$

converges for $\beta \geq \beta_c$, then for $(C^*(S), \mathbb{R}, \sigma)$ we have

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1. *$\beta \in [0, 1)$: no KMS_β state;*
2. *$\beta \in [1, \beta_c]$: unique KMS_β if action $S_c \curvearrowright S/S_c$ essentially free, where $S_c \subset S$ subsemigroup of elements having LCM with any t in S;*

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3. *$\beta \in (\beta_c, \infty]$: KMS_β states parametrised by normalised traces on $C^*(S_c)$;*

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3. *$\beta \in (\beta_c, \infty]$: KMS_β states parametrised by normalised traces on $C^*(S_c)$;*
4. *Ground states: parametrised by states on $C^*(S_c)$.*