Topological Full Groups and Graph Groupoids

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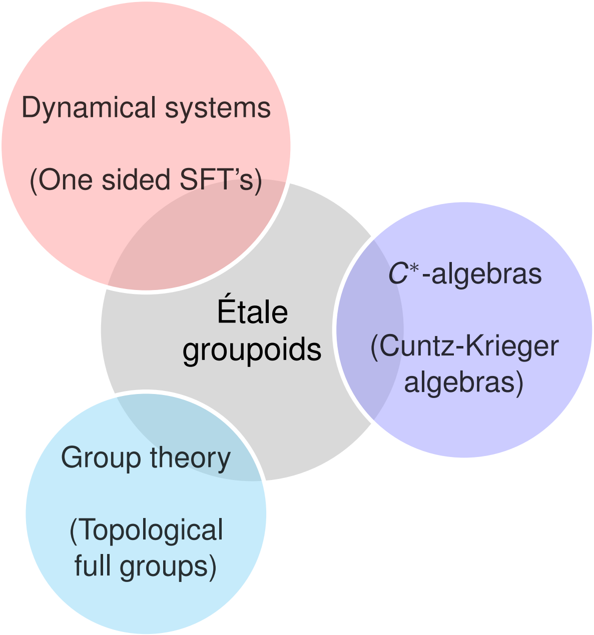
Work in progress with Eduard Ortega

Topological Full Groups in the Literature

- Measurable dynamics (Dye '63)
- Topological dynamics (Giordano-Putnam-Skau '99, Matsumoto '10)
- Étale groupoids (Matui '11)
- Group theory (Grigorchuk-Medynets & Juschenko-Monod '12)
- Inverse semigroups (Lawson '17)

Motivation


Dynamical systems
(One sided SFT's)
Étale groupoids
C^*-algebras
(Cuntz-Krieger algebras)
Group theory
(Topological full groups)

Groupoids

A topological groupoid $\mathcal{G}$ has:

- partially defined multiplication (continuous)
- inverses g^{-1} (continuous)
- right units $s(g) = g^{-1}g$, left units $r(g) = gg^{-1}$
- $\mathcal{G}^{(0)} = \{g^{-1}g \mid g \in \mathcal{G}\}$ unit space
- $\mathcal{G}$ étale if $s, r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ local homeomorphisms
- $\mathrm{Iso}(x) = \{x \xrightarrow{g} x\}$
- $\mathcal{G}$ effective if $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$
- $\mathrm{orbit}_{\mathcal{G}}(x) = \{y \in \mathcal{G}^{(0)} \mid x \xrightarrow{g} y\}$
- $\mathcal{G}$ minimal if every orbit is dense in $\mathcal{G}^{(0)}$

Assume $\mathcal{G}$ étale, Hausdorff, second countable, locally compact (ample / $\mathcal{G}^{(0)}$ totally disconnected)

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Groupoids

Example (Transformation groupoid)

- $\Gamma \curvearrowright X$ group action
- $X \rtimes \Gamma = X \times \Gamma$
- $x \xrightarrow{(x,\gamma)} \gamma \cdot x$
- $(X \rtimes \Gamma)^{(0)} = X \times \{e\} = X$

Example (Groupoid of germs)

- $\mathrm{Germ}(\Gamma \curvearrowright X) = (X \rtimes \Gamma)/{\sim}$
- $(x, \gamma) \sim (x, \gamma') \iff \gamma, \gamma'$ agree on a n'hood of x
- NB: not always Hausdorff

Groupoids

Example (Deaconu-Renault groupoids)

- $\sigma \colon X \to X$ partial local homeomorphism
- $\mathcal{G}(X, \sigma) = \{(x, m-n, y) \mid \sigma^m(x) = \sigma^n(y)\}$
- $y \xrightarrow{(x,m-n,y)} x$
- $\mathcal{G}_\sigma^{(0)} = \{(x, 0, x) \mid x \in X\} = X$

Example (Graph groupoid)

- E directed graph
- $\partial E = E^\infty \cup E^*_{\text{sing}}, \quad \sigma_E \colon \partial E \to \partial E$ shift map
- $\mathcal{G}_E = \mathcal{G}(\partial E, \sigma_E)$
- $\mathcal{G}_E^{(0)} = \partial E$ compact $\iff$ finitely many vertices

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The Topological Full Group

$U \subseteq \mathcal{G}$ open bisection ($s_{|U}, r_{|U}$ injective)

$$\pi_U \colon s(U) \longrightarrow r(U) \text{ homeomorphism}$$
$$s(g) \longmapsto r(g), \quad g \in U$$

$$U \text{ full} \implies \pi_U \colon \mathcal{G}^{(0)} \longrightarrow \mathcal{G}^{(0)}$$

Definition (Matui '11)

$\mathcal{G}$ étale and $\mathcal{G}^{(0)}$ compact.

- $[\![\mathcal{G}]\!] := \{\pi_U \mid U \subseteq \mathcal{G}$ compact open full bisection$\}$

- $[\![\mathcal{G}]\!] \leq \text{Homeo}\left(\mathcal{G}^{(0)}\right)$
- $\pi_U \circ \pi_V = \pi_{UV}$
- $\text{id}_{\mathcal{G}^{(0)}} = \pi_{\mathcal{G}^{(0)}}$
- $(\pi_U)^{-1} = \pi_{U^{-1}}$

The Topological Full Group

- $[\![\mathcal{G}]\!]$ = "finitary" homeomorphisms of $\mathcal{G}^{(0)}$ (which preserve orbits in a continuous manner)

Example (Transformation groupoid)

- $T\colon X \xrightarrow{\simeq} X$ (X compact)
- $[\![X \rtimes_T \mathbb{Z}]\!] = \{\pi\colon X \xrightarrow{\simeq} X \mid \pi(x) = T^{k(x)}(x),$
  $k\colon X \to \mathbb{Z}$ continuous$\}$
- "finitely many piecewise powers of T"

Example (Deaconu-Renault groupoids)

- $[\![\mathcal{G}_\sigma]\!] = \{\pi\colon X \xrightarrow{\simeq} X \mid \sigma^{m(x)}(\pi(x)) = \sigma^{n(x)}(x),$
  (X compact) $m, n\colon X \to \mathbb{N}_0$ continuous$\}$
- $[\![\mathcal{G}_E]\!]$ "change bounded prefix of infinite paths": $\mu y \mapsto \nu y$

Matui's Spatial Realization Theorem

Theorem (Matui '15)

$\mathcal{G}, \mathcal{H}$ effective minimal étale groupoids with $\mathcal{G}^{(0)}, \mathcal{H}^{(0)}$ Cantor spaces, then TFAE:

1. $\mathcal{G} \cong \mathcal{H}$
2. $[\![\mathcal{G}]\!] \cong [\![\mathcal{H}]\!]$
3. $D([\![\mathcal{G}]\!]) \cong D([\![\mathcal{H}]\!])$

- Also true for other subgroups (Nekrashevych '17)

Extending the Topological Full Group

- $\pi\colon X \to X, \quad \mathrm{supp}(\pi) = \overline{\{x \in X \mid \pi(x) \neq x\}}$

Definition

$\mathcal{G}$ effective étale and $\mathcal{G}^{(0)}$ locally compact.

- $[\![\mathcal{G}]\!] := \{\pi_U \mid U \text{ clopen full bisection, } \mathrm{supp}(\pi_U) \text{ compact}\}$

- U as above $\implies$ $U = V \sqcup A$, V compact open bisection with $s(V) = r(V)$ and $A \subseteq \mathcal{G}^{(0)}$ clopen
- $\mathcal{G}$ second countable $\implies$ $[\![\mathcal{G}]\!]$ countable

Strategy for Reconstruction

$$[\![\mathcal{G}]\!] \cong [\![\mathcal{H}]\!]$$

$$\Big\Downarrow \text{ Rubin's Theorem}$$

$$(\mathcal{G}^{(0)}, [\![\mathcal{G}]\!]) \cong (\mathcal{H}^{(0)}, [\![\mathcal{H}]\!]) \text{ spatially}$$

$$\Big\Downarrow \text{ Functoriality}$$

$$\mathrm{Germ}([\![\mathcal{G}]\!] \curvearrowright \mathcal{G}^{(0)}) \cong \mathrm{Germ}([\![\mathcal{H}]\!] \curvearrowright \mathcal{H}^{(0)})$$

$$\Big\Downarrow \ \mathcal{G},\mathcal{H} \text{ covered by full open bisections}$$

$$\mathrm{Germ}([\![\mathcal{G}]\!] \curvearrowright \mathcal{G}^{(0)}) \cong \mathcal{G} \cong \mathcal{H} \cong \mathrm{Germ}([\![\mathcal{H}]\!] \curvearrowright \mathcal{H}^{(0)})$$

One of Rubin's Many Theorems

Definition

$\Gamma \leq \mathrm{Homeo}(X)$ is locally moving if

- $\emptyset \neq A \subseteq X$ open $\implies \exists\, \gamma \in \Gamma \setminus \{\mathrm{id}_X\}$ with $\mathrm{supp}(\gamma) \subseteq A$.

- Γ locally moving $\implies$ X has no isolated points

Theorem (Rubin '89)

X_1, X_2 Hausdorff, locally compact, totally disconnected and first countable with

- $\Gamma_i \leq \mathrm{Homeo}(X_i)$ *locally moving & flexible*
- $|\Gamma_i(x)| \geq 3\ \forall\, x \in X_i$

Then any (abstract) isomorphism $\Phi\colon \Gamma_1 \to \Gamma_2$ is spatial, i.e.

$$\exists\, h\colon X_1 \xrightarrow{\cong} X_2 \text{ such that } \Phi(\gamma) = h \circ \gamma \circ h^{-1}\ \forall\, \gamma \in \Gamma_1$$

Our Reconstruction Theorem

Theorem

$\mathcal{G}, \mathcal{H}$ effective étale groupoids with $\mathcal{G}^{(0)}, \mathcal{H}^{(0)}$ locally compact, totally disconnected (& no isolated points) and

- $[\![\mathcal{G}]\!], [\![\mathcal{H}]\!]$ *locally moving*
- $\mathcal{G}$*-orbits,* $\mathcal{H}$*-orbits* ≥ 3

Then TFAE:

1. $\mathcal{G} \cong \mathcal{H}$
2. $[\![\mathcal{G}]\!] \cong [\![\mathcal{H}]\!]$

Application to Graph Groupoids & C^*-algebras

E countable graph

- condition $(I) \implies$ condition $(L) \iff \mathcal{G}_E$ effective
- no sinks & condition $(I) \implies [\![\mathcal{G}_E]\!]$ locally moving
- no sinks & no "bad sources" $\implies$ orbits ≥ 3

Application to Graph Groupoids & C^*-algebras

Corollary

E, F countable graphs with no sinks, no "bad sources", and satisfying condition (I). Then TFAE:

1. $C^*(E)$ *and* $C^*(F)$ *are isomorphic as* C^**-algebras, s.t.* $\mathcal{D}(E) \longmapsto \mathcal{D}(F)$.
2. $\mathcal{G}_E$ *and* $\mathcal{G}_F$ *are isomorphic as topological groupoids.*
3. $[\![\mathcal{G}_E]\!]$ *and* $[\![\mathcal{G}_F]\!]$ *are isomorphic as abstract groups.*
4. E *and* F *are orbit equivalent graphs.*

- $1 \Leftrightarrow 2$: Renault '08
- $2 \Leftrightarrow 4$: Brownlowe, Carlsen, Whittaker '17

Thanks for listening!