Automorphisms of Kirchberg algebras arising from groupoids and inverse semigroups

Facets of Irreversibility: Inverse Semigroups, Groupoids, and Operator Algebras
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Definition (Cuntz 1981)

A simple C^*-algebra A is called purely infinite if for all non-zero $a, b \in A$ there exist $x, y \in A$ such $xay = b$.

A separable, simple, purely infinite, nuclear C^*-algebra is said to be a Kirchberg algebra.

Theorem (Cuntz 1977 & 1981)

The C^*-algebras $\mathcal{O}_n$ are unital Kirchberg algebras. Moreover, their K-theory is given by $K_0(\mathcal{O}_n) = C_{n-1}[1]$ and $K_1(\mathcal{O}_n) = 0$.

Among unital Kirchberg algebras, the Cuntz algebras are "almost" characterized by their K-theoretic properties.

Definition (Rosenberg-Schochet 1987)

A separable C^*-algebra A is said to satisfy the *universal coefficient theorem* (UCT) if for every separable C^*-algebra B the following sequence is exact

$$\mathrm{Ext}^1(K_*(A), K_{1-*}(B)) \rightarrowtail KK(A,B) \twoheadrightarrow \mathrm{Hom}(K_*(A), K_*(B)).$$

A separable C^*-algebra satisfies the UCT if and only if it is KK-equivalent to an abelian C^*-algebra.
Moreover, the class of separable C^*-algebras satisfying the UCT is large and has good permanence properties.

Question (UCT problem)

Does every separable, nuclear C^*-algebra satisfy the UCT?

Theorem (Kirchberg 1995)

The UCT problem is equivalent to the question whether every unital Kirchberg algebra satisfies the UCT.

Theorem (Kirchberg-Phillips classification)

Let A, B be Kirchberg algebras. Then A and B are stably isomorphic if and only if they are KK-equivalent.
In particular, if A and B satisfy the UCT then they are stably isomorphic if and only if they are isomorphic in K-theory.

Theorem (Spielberg 2007, Katsura 2008 & Yeend 2006 & 2007)

Every UCT Kirchberg algebra is the (reduced) C^*-algebra of an ample, locally compact, Hausdorff, second countable, topologically principal groupoid.

Combining this with a result by Exel (2010), it follows that every unital UCT Kirchberg algebra is the tight C^*-algebra of an inverse semigroup.

One may ask to what extend this is true on the level of $*$-automorphisms as well.

Question

Let A be a UCT Kirchberg algebra and let $\alpha \in \mathrm{Aut}(A)$ be a $*$-automorphism. When does there exist an ample, locally compact, Hausdorff, second countable, topologically principal groupoid $\mathcal{G}$ and a continuous groupoid automorphism φ of $\mathcal{G}$ such that $(A, \alpha) \cong (\mathrm{C}^*_{\mathrm{red}}(\mathcal{G}), \varphi^*)$?

Similarly, one might ask for the existence of an inverse semigroup automorphism to induce the given $*$-automorphism on the level of the associated tight C^*-algebra.

We will give a partial answer to this for certain finite order automorphisms and also link this to the UCT problem.

Definition (Izumi 2004)

Let G be a finite group and A a separable, unital C^*-algebra. An action $\alpha : G \curvearrowright A$ is said to have the Rokhlin property if for all $\varepsilon > 0$ and for every finite set $\mathcal{F} \subset A$ there exists a partition of unity by projections $\{p_g \ : \ g \in G\} \subset A$ such that

(i) $\alpha_g(p_h) = p_{gh}$ for all $g, h \in G$;

(ii) $\|p_g a - a p_g\| < \varepsilon$ for all $g \in G$ and $a \in \mathcal{F}$.

Actions with the Rokhlin property have many several desirable features. For example, many C^*-algebraic properties pass from the coefficient algebra to the corresponding crossed product.

There are K-theoretical restrictions to the existence of actions with the Rokhlin property. For example, there is no Rokhlin action of a non-trivial, finite group on $\mathcal{O}_\infty$.

The Rokhlin property also enjoys a certain rigidity with respect to approximate unitary equivalence.

Theorem (Izumi 2004)

Let G be a finite group and A a separable, unital C^*-algebra. Assume that $\alpha, \beta : G \curvearrowright A$ are two actions with the Rokhlin property. If for all $g \in G$, α_g and β_g are approximately unitarily equivalent, then α and β are conjugate (via an approximately inner automorphism).

As a consequence, one obtains that up to conjugacy there exists for every finite group a unique action with the Rokhlin property on $\mathcal{O}_2$.

Another important feature of the Rokhlin property is its duality with another class of actions in the case of finite abelian groups.

Definition (Izumi 2004)

Let G be a finite, abelian group and A a separable, unital C^*-algebra. An action $\alpha : G \curvearrowright A$ is said to be

(i) strongly approximately inner if there exist unitaries $u_{n,g} \in A^\alpha$, $n \in \mathbb{N}$ and $g \in G$, such that $\alpha_g = \lim_{n\to\infty} \mathrm{Ad}(u_{n,g})$ for all $g \in G$;

(ii) approximately representable if it is strongly approximately inner and the unitaries $u_{n,g}$ can be chosen such that $g \mapsto u_{n,g}$ defines an (approximate) unitary representation for all $n \in \mathbb{N}$.

In general, there is a difference between strong approximate innerness and approximate representability.
A result by Izumi (2004) however states that for outer actions of $\mathbb{Z}_{p^k}$ on $\mathcal{O}_2$, where p is prime and $k \geq 1$, these notions coincide.
It is open whether all such actions on $\mathcal{O}_2$ are strongly approximately inner.

Proposition (Izumi 2004)

Let G be a finite, abelian group and A a separable, unital C^*-algebra. Let $\alpha : G \curvearrowright A$ be an action. Then

(i) α has the Rokhlin property if and only if $\hat{\alpha}$ is approximately representable;

(ii) α is approximately representable if and only if $\hat{\alpha}$ has the Rokhlin property.

Let us indicate how to prove the "only if" direction of (i).
Given a set of Rokhlin projections $\{p_g \ : \ g \in G\} \subset A$ for α, we set for $\chi \in \hat{G}$

$$u_\chi = \sum_{g \in G} \chi(g) p_g \in \mathcal{U}(A).$$

Such unitaries approximately implement $\hat{\alpha}$ and thus witness approximate representability for $\hat{\alpha}$ as $A = (A \rtimes_\alpha G)^{\hat{\alpha}}$.

Using the rigidity of Rokhlin action, Izumi was able to classify outer strongly approximately inner actions on $\mathcal{O}_2$ of cyclic groups of prime power order. In the case of $\mathbb{Z}_2$-actions, the result is particularly appealing.

Theorem (Izumi 2004)

Let $\alpha, \beta : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2$ be two outer strongly approximately inner actions. Then α and β are (cocycle) conjugate if and only if their fixed point algebras $\mathcal{O}_2^\alpha$ and $\mathcal{O}_2^\beta$ are (stably) isomorphic.

As shown by Izumi (2004) or B.-Szabó (2017), many Kirchberg algebras (absorbing a suitable UHF algebra) can be written as crossed products of $\mathcal{O}_2$ by outer approximately representable actions of finite cyclic groups.

Theorem (B.-Szabó 2017)

The UCT problem is equivalent to the question whether all crossed products of the form $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ satisfy the UCT, where $p \geq 2$ is prime and α is outer approximately representable.

Definition (Renault 2008)

A C^*-subalgebra B of a C^*-algebra A is called a Cartan subalgebra if

(i) B contains an approximate unit for A;

(ii) B is a maximal abelian $*$-subalgebra;

(iii) $\mathrm{C}^*(\{a \in A \ : \ aBa^* \subseteq B \text{ and } a^*Ba \subseteq B\}) = A$;

(iv) there exists a faithful conditional expectation $A \to B$.

(A, B) is called a Cartan pair.

Theorem (Renault 2008)

Cartan pairs (A, B), with A separable, are exactly of the form $(\mathrm{C}^*_{\mathrm{red}}(\mathcal{G}, \Sigma), C_0(\mathcal{G}^{(0)}))$ for some twisted étale, locally compact, second countable, Hausdorff, topologically principal groupoid $(\mathcal{G}, \Sigma)$.

Given a $*$-automorphism $\alpha \in \mathrm{Aut}(A)$, an inverse semigroup $\mathcal{S} \subset A$ is said to be α-homogeneous if for every $s \in \mathcal{S}$ there is some $\lambda \in \mathbb{C}$ such that $\alpha(s) = \lambda s$.

Theorem (B.-Li)

Let A be a unital Kirchberg algebra satisfying the UCT. Assume that $\alpha : \mathbb{Z}_n \curvearrowright A$ is an outer approximately representable action such that $A \rtimes_\alpha \mathbb{Z}_n \cong (A \rtimes_\alpha \mathbb{Z}_n) \otimes M_{n^\infty}$. Then the following are equivalent:

(i) *$A \rtimes_\alpha \mathbb{Z}_n$ satisfies the UCT;*

(ii) *there exists an α-homogeneous inverse semigroup $\mathcal{S} \subset A$ such that $A = \mathrm{C}^*(\mathcal{S})$ and $\mathrm{C}^*(E(\mathcal{S}))$ is a Cartan subalgebra in both A^α and A;*

(iii) *there exists a Cartan subalgebra $C \subset A$ such that $\alpha(C) = C$.*

In (ii), we may actually assume that A is (isomorphic to) the tight C^*-algebra of $\mathcal{S}$.

Using Renault's characterization, α in (iii) is induced by an automorphism of a twisted groupoid $(\mathcal{G}, \Sigma)$ such that $(A, C) \cong (\mathrm{C}^*_{\mathrm{red}}(\mathcal{G}, \Sigma), C(\mathcal{G}^{(0)}))$.

Main ideas of the proof of $(i) \Rightarrow (ii)$:

Combining results of Spielberg (2007) and Katsura (2008), we find an ample, locally compact, Hausdorff, topologically principal groupoid $\mathcal{G}$ and an action $\varphi : \mathbb{Z}_n \curvearrowright \mathcal{G}$ such that

(i) $\mathrm{C}^*_{\mathrm{red}}(\mathcal{G}) \cong A \rtimes_\alpha \mathbb{Z}_n$;
(ii) $K_*(\varphi^*) = K_*(\hat{\alpha})$.

As α is approximately representable, $\hat{\alpha} : \mathbb{Z}_n \curvearrowright A \rtimes_\alpha \mathbb{Z}_n$ has the Rokhlin property.
Using φ, Izumi's rigidity for Rokhlin actions, and a model action result for actions with the Rokhlin property (B.-Szabó (2017)), we can show that

1) $\hat{\alpha} : \mathbb{Z}_n \curvearrowright A \rtimes_\alpha \mathbb{Z}_n$ and
2) $\hat{\hat{\alpha}} : \mathbb{Z}_n \curvearrowright A \rtimes_\alpha \mathbb{Z}_n \rtimes_{\hat{\alpha}} \mathbb{Z}_n$

are induced by actions on ample, locally compact, Hausdorff, topologically principal groupoids as well.
By Takai duality, $(A \rtimes_\alpha \mathbb{Z}_n \rtimes_\alpha \mathbb{Z}_n, \hat{\hat{\alpha}}) \cong (A \otimes M_n, \alpha \otimes \rho)$, and some extra work yields $(i) \Rightarrow (ii)$.

The theorem in particular applies if $\alpha : \mathbb{Z}_n \curvearrowright \mathcal{O}_2$ is outer approximately representable and $n = p^k$ for some prime $p \geq 2$ and some $k \geq 1$.

Using this, one can deduce the following characterization of the UCT problem.

Theorem (B.-Li)

The following statements are equivalent:

(i) *every separable, nuclear C^*-algebra satisfies the UCT;*

(ii) *for every prime number $p \geq 2$ and every outer approximately representable action $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ there exists an α-homogeneous inverse semigroup $\mathcal{S} \subset A$ such that $\mathcal{O}_2 = \mathrm{C}^*(\mathcal{S})$ and $\mathrm{C}^*(E(\mathcal{S}))$ is a Cartan subalgebra in both $\mathcal{O}_2^\alpha$ and $\mathcal{O}_2$;*

(iii) *every outer approximately representable $\mathbb{Z}_p$-action on $\mathcal{O}_2$ with $p = 2$ or $p = 3$ fixes some Cartan subalgebra $C \subset \mathcal{O}_2$ globally.*

We have seen so far that (certain) approximately representable actions of finite cyclic groups on UCT Kirchberg algebras come from combinatorial or geometric data, provided that the crossed products satisfy the UCT.

One now might ask to what extend the converse holds, that is, approximate representability is automatic for such actions.

Question

Let $\alpha : \mathbb{Z}_n \curvearrowright A$ an outer action on a unital UCT Kirchberg algebra. Assume that there exists an α-homogeneous inverse semigroup $\mathcal{S} \subset A$ satisfying

(i) $\mathrm{C}^*(\mathcal{S}) = A$;

(ii) $\mathrm{C}^*(E(\mathcal{S})) \subset A$ is a Cartan subalgebra.

When is α strongly approximately inner/approximately representable?

In this situation, $A \rtimes_\alpha \mathbb{Z}_n$ always satisfies the UCT (B.-Li (2017)).

We give a partial answer in the case of the canonical inverse semigroups associated with Cuntz-Krieger algebras.

Definition (Cuntz-Krieger 1980)

Let $A \in M_n(\{0,1\})$ be a matrix with no zero rows or columns. The Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by a family of partial isometries $\{T_1, \ldots, T_n\}$ subject to the relations

(i) $T_i^* T_j = 0$ if $i \neq j$;

(ii) $T_i^* T_i = \sum_{j=1}^{n} A(i,j) T_j T_j^*$.

The canonical partial isometries T_i generate an inverse semigroup $\mathcal{S}_A$, which in turn generates $\mathcal{O}_A$ as a C^*-algebra.

The idempotent lattice associated with $\mathcal{S}_A$ generates a commutative C^*-subalgebra $\mathcal{D}_A \subset \mathcal{O}_A$, which is a Cartan subalgebra if $\mathcal{O}_A$ is simple.

Automorphisms of $\mathcal{O}_A$ for which $\mathcal{S}_A$ is homogeneous are quasi-free in the sense of Zacharias (2000) (or Evans (1980) for Cuntz algebras in their standard presentation).

Theorem (Izumi 2004)

Let $2 \leq n < \infty$. Every action α of a finite abelian group G on $\mathcal{O}_n$ by quasi-free automorphisms is strongly approximately inner. Moreover, if $G = \mathbb{Z}_{p^k}$ for some prime number p with $n \not\equiv 1 (mod\ p)$ and some $k \geq 1$, then α is approximately representable.

Example (Izumi 2004)

The quasi-free action $\alpha : \mathbb{Z}_2 \curvearrowright \mathcal{O}_3$ given by $\alpha(S_1) = S_1$, $\alpha(S_2) = S_2$ and $\alpha(S_3) = -S_3$ is not approximately representable.

Theorem (follows from Cuntz-Krieger 1980)

Let $A \in M_n(\{0,1\})$ be an aperiodic matrix, that is, there exists $k \geq 1$ such that all entries of A^k are strictly positive. Then $\mathcal{O}_A$ is simple and purely infinite. In fact, in this situation $\mathcal{O}_A$ is a unital UCT Kirchberg algebra.

Theorem (B.-Szabó)

Let G be a finite abelian group and let $A \in M_n(\{0,1\})$ be an aperiodic matrix. Let $\sigma : G \curvearrowright \mathcal{O}_A$ be an action for which $\mathcal{S}_A$ is homogeneous. If σ is outer, then it is strongly approximately inner.

If G is cyclic of prime power order and $\mathcal{O}_A$ is (possibly non-canonically) isomorphic to $\mathcal{O}_2$, we also get approximate representability for such actions.
In this situation, Izumi's classification applies.
Combining this with Kirchberg-Phillips classification, the case $G = \mathbb{Z}_2$ reads as follows.

Corollary

Let $m, n \geq 2$ and let $A \in M_m(\{0,1\})$ and $B \in M_n(\{0,1\})$ be aperiodic matrices. Assume that $\mathcal{O}_A \cong \mathcal{O}_2 \cong \mathcal{O}_B$. Let $\alpha : \mathbb{Z}_2 \curvearrowright \mathcal{O}_A$ and $\beta : \mathbb{Z}_2 \curvearrowright \mathcal{O}_B$ be outer actions for which $\mathcal{S}_A$ and $\mathcal{S}_B$ are homogeneous, respectively.
Then α and β are (cocycle) conjugate if and only if their fixed point algebras $\mathcal{O}_A^\alpha$ and $\mathcal{O}_B^\beta$ are (stably) isomorphic.

Thank you for your attention!