Theorem 1

Let $A$ and $B$ be finite square $\{0, 1\}$-matrices with no zero rows and no zero columns, and let $R$ be an indecomposable reduced commutative ring with unit. The following are equivalent.

1. The one-sided shift spaces $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent.
2. The groupoids $G_A$ and $G_B$ are isomorphic as topological groupoids.
3. The inverse semigroups $S_A$ and $S_B$ are isomorphic.
4. The Cuntz–Krieger algebras $O_A$ and $O_B$ are isomorphic by a diagonal-preserving isomorphism.
5. The Steinberg algebras $RG_A$ and $RG_B$ are isomorphic by a diagonal-preserving isomorphism.
One-sided shifts of finite type

Let $A \in M_N(\{0, 1\})$ be a matrix with no zero rows and no zero columns. The space

$$X_A := \{(x_n)_{n \in \mathbb{N}} \in \{1, 2, \ldots, N\}^\mathbb{N} : A_{x_n, x_{n+1}} = 1 \text{ for all } n\}$$

is a totally disconnected compact Hausdorff space. The map $\sigma_A : X_A \to X_A$

$$x_0x_1x_2 \cdots \mapsto x_1x_2\ldots$$

is a local homeomorphism. The pair $(X_A, \sigma_A)$ is a shift of finite type.

The orbit of $x \in X_A$ is

$$\bigcup_{m, n \in \mathbb{N}} \sigma_A^{-n}(\{x\}) = \{y \in X_A : \sigma_A^k(x) = \sigma_A^l(y) \text{ for some } k, l \in \mathbb{N}\}.$$ 

Continuous orbit equivalence

Let $A$ and $B$ be finite square $\{0, 1\}$-matrices with no zero rows and no zero columns. If $h : X_A \to X_B$ is a homeomorphism that maps orbits to orbits, then for each $x \in X_A$ there exist nonnegative integers $k_x, l_x$ such that

$$\sigma_B^{k_x}(h(\sigma_A(x))) = \sigma_B^{l_x}(h(x)).$$

We say that $h$ is a continuous orbit equivalence and that $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent if there are continuous functions $k, l : X_A \to \mathbb{N}$ and $k', l' : X_B \to \mathbb{N}$ such that

$$\sigma_B^{k(x)}(h(\sigma_A(x))) = \sigma_B^{l(x)}(h(x)) \quad \sigma_A^{k'(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l'(y)}(h^{-1}(y))$$

for all $x \in X_A$ and all $y \in X_B$.
This notion was, in this setting, introduced by Matsumoto.
The groupoid $G_A$

The *Deaconu-Renault groupoid* associated to the one-sided shift of finite type $(X_A, \sigma_A)$ is

$$G_A := \{(x, n, y) \in X_A \times \mathbb{Z} \times X_A \mid \exists k, l \in \mathbb{N}, \ n = k - l: \sigma_A^k(x) = \sigma_A^l(y)\}$$

with unit space $G_A^{(0)} = \{(x, 0, x) \in G_A \mid x \in X_A\}$. The range map is $r(x, n, y) = (x, 0, x)$ and the source map is $s(x, n, y) = (y, 0, y)$. The product $(x, n, y)(x', n', y')$ is well-defined if and only if $y = x'$ in which case it equals $(x, n + n', y')$, while inversion is given by $(x, n, y)^{-1} = (y, -n, x)$. The topology on $G_A$ has a basis consisting of sets of the form

$$Z(U, k, l, V) := \{(x, k - l, y) \in G_A \mid x \in U, y \in V\},$$

where $k, l \in \mathbb{N}$ and $U, V \subseteq X_A$ are open such that $\sigma_A^k|_U$ and $\sigma_A^l|_V$ are injective and $\sigma_A^k(U) = \sigma_A^l(V)$. With this, $G_A$ is an amenable, second countable, étale, and locally compact Hausdorff groupoid.

Continuous orbit equivalence and groupoid isomorphism

If $h : X_A \to X_B$ is a continuous orbit equivalence, then there exists a function $m : G_A \to \mathbb{Z}$ such that

$$(x, n, y) \mapsto (h(x), m(x, n, y), h(y))$$

is an isomorphism between $G_A$ and $G_B$ (one have to take a little care if there are isolated periodic points).

Conversely, if $\phi : G_A \to G_B$ is an isomorphism, then $\phi$ restricts to a homeomorphism between $G_A^{(0)}$ and $G_B^{(0)}$ and thus induces a homeomorphism $h : X_A \to X_B$ such that

$$\phi(x, 0, x) = (h(x), 0, h(x))$$

for $x \in X_A$, and such that $h$ is a continuous orbit equivalence.
The inverse semigroup $S_A$

The inverse semigroup $S_A$ consists of homeomorphisms $\eta : \text{dom}(\eta) \to \text{ran}(\eta)$ where $\text{dom}(\eta) = U \times \mathbb{Z}$ and $\text{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets $U, V$ of $X_A$, and there are continuous functions $k : U \to \mathbb{N}$ and $l : V \to \mathbb{N}$ such that if $\eta(x, m) = (y, n)$, then $\sigma_{A}^{k(x)}(x) = \sigma_{A}^{l(y)}(y)$ and $m + l(y) = n + k(x)$. A continuous orbit equivalence $h : X_A \to X_B$ induces an isomorphism $\psi : S_A \to S_B$ such that if $\eta \in S_A$ and $\text{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), m + k)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$ and $k$ is some integer that depends on $x$ and $\eta$.

The set of characters of the idempotent semi-lattice $\mathcal{E}(S_A)$ of $S_A$ is homeomorphic to $X_A$, and the groupoid of germs of the action of $S_A$ on the set of characters of $\mathcal{E}(S_A)$ is isomorphic to $G_A$. It follows that an isomorphism between $S_A$ and $S_B$ induces an isomorphism between $G_A$ and $G_B$.

Cuntz–Krieger algebras

The Cuntz–Krieger algebra $O_A$ of $A$ is the universal unital $C^*$-algebra generated by partial isometries $s_1, \ldots, s_N$ subject to the conditions

$$s_i^* s_j = 0 \ (i \neq j), \quad s_i^* s_i = \sum_{j=1}^{N} A_{ij} s_j s_j^*$$

for every $i = 1, \ldots, N$.

If $\alpha = \alpha_1 \ldots \alpha_n$ is a word of elements from $\{1, \ldots, N\}$, then we let $s_\alpha := s_{\alpha_1} \ldots s_{\alpha_n}$, and we let

$$\mathcal{D}_A := \overline{\text{span}} \{ s_\alpha s_\alpha^* : \alpha \text{ is a finite word} \}.$$ 

Then $\mathcal{D}_A$ is isomorphic to $C(X_A)$.
Cuntz–Krieger algebras

There is an isomorphism from $C^*(G_A)$ to $\mathcal{O}_A$ that maps $C(G_A^{(0)})$ onto $\mathcal{D}_A$. It follows that if $G_A$ and $G_B$ are isomorphic, then there is an isomorphism $\phi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\phi(\mathcal{D}_A) = \mathcal{D}_B$. Conversely, if there is an isomorphism $\phi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\phi(\mathcal{D}_A) = \mathcal{D}_B$, then $G_A$ and $G_B$ are isomorphic. If $A$ and $B$ both satisfy the condition introduced by Cuntz and Krieger called condition (I), then this follows from a result by Renault which says that two second countable locally compact topological principal étale Hausdorff groupoids are isomorphic if and only if there is a diagonal-preserving isomorphism between their reduced $C^*$-algebras. The general case follows from a result by Brownlowe–C–Whittaker that says the groupoids of two countable directed graphs are isomorphic if and only if there is a diagonal-preserving isomorphism between their $C^*$-algebras.

Reconstruction of groupoids

This has recently been generalised by C–Ruiz-Sims-Tomforde that show that two second countable locally compact étale Hausdorff groupoids for which the interior of the isotropi is abelian and torsion-free, are isomorphic if and only if there is a diagonal-preserving isomorphism between their reduced $C^*$-algebras. The equivalence of 1 and 4 was also proven by Matsumoto for irreducible matrices satisfying condition (I) without the use of groupoids.
Steinberg algebras

The Steinberg algebra $RG_A$ of $G_A$ is $C_c(G_A, R)$ equipped with a convolution product defined similar to how the product of $C^*(G_A)$ is defined. If $R = \mathbb{C}$, then $RG_A$ is a dense subalgebra of $C^*(G_A)$.

It follows that if $G_A$ and $G_B$ are isomorphic, then there is an isomorphism from $RG_A$ to $RG_B$ that maps $RG_A^{(0)}$ onto $RG_B^{(0)}$.

The converse follows from a recent result of Steinberg which says that if $R$ is an indecomposable commutative ring with unit, and $G_1$ and $G_2$ are Hausdoff ample groupoids such that $G_1$ has a dense set of objects $x$ such that the group algebra over $R$ of the isotropy group at $x$ of the interior of the isotropy bundle of $G_1$ has no non-trivial units, then $G_1$ and $G_2$ are isomorphic if and only if their is a diagonal-preserving isomorphism between $RG_1$ and $RG_2$.

Theorem 2

Let $A$ and $B$ be finite square $\{0, 1\}$-matrices with no zero rows and no zero columns, and let $R$ be an indecomposable commutative ring with unit. The following are equivalent.

1. The one-sided shift spaces $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are eventually conjugate.
2. There is an isomorphism $\phi : G_A \to G_B$ such that $c_A = c_B \circ \phi$.
3. There is an isomorphism $\psi : S^e_A \to S^e_B$ such that $e_A = e_B \circ \psi$.
4. The Cuntz–Krieger algebras $O_A$ and $O_B$ are isomorphic by a diagonal-preserving isomorphism that intertwines the gauge actions $\lambda^A_t$ and $\lambda^B_t$.
5. The Steinberg algebras $RG_A$ and $RG_B$ are isomorphic by a graded diagonal-preserving isomorphism.
Eventually conjugacy

Let $h : X_A \to X_B$ be a homeomorphism. We say that $h$ is an eventual conjugacy and that $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are eventually conjugate if there is an $k \in \mathbb{N}$ such that

$$\sigma_B^k(h(\sigma_A(x))) = \sigma_B^{k+1}(h(x)) \quad \sigma_A^k(h^{-1}(\sigma_B(y))) = \sigma_A^{k+1}(h^{-1}(y))$$

for all $x \in X_A$ and all $y \in X_B$. This is a notion introduced by Matsumoto.

The cocycle $c_A$

The function $c_A : G_A \to \mathbb{Z}$ is defined by $c_A(x, n, y) = n$. This is a continous cocycle in the sense $c_A((x, n, y)^{-1}) = -c_A(x, n, y)$ and $c_A((x, n, y)(y, m, z)) = c_A(x, n + m, z)$.

If $h : X_A \to X_B$ is an eventual conjugacy, then $(x, n, y) \mapsto (h(x), n, h(y))$ is an isomorphism between $G_A$ and $G_B$ such that $c_A = c_B \circ \phi$.

Conversely, if $\phi : G_A \to G_B$ is an isomorphism such that $c_A = c_B \circ \phi$, then $\phi$ restricts to a homeomorphism between $G_A^{(0)}$ and $G_B^{(0)}$ and thus induces a homeomorphism $h : X_A \to X_B$ such that $\phi(x, 0, x) = (h(x), 0, h(x))$ for $x \in X_A$, and such that $h$ is an eventual conjugacy.
The inverse semigroup $S^e_A$ and the homomorphism $e_A : S^e_A \rightarrow \mathbb{Z}$

The inverse semigroup $S^e_A$ is the subsemigroup of $S_A$ consisting of homeomorphisms $\eta : \text{dom}(\eta) \rightarrow \text{ran}(\eta)$ where $\text{dom}(\eta) = U \times \mathbb{Z}$ and $\text{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets $U, V$ of $X_A$, and there are $k, l \in \mathbb{N}$ such that if $\eta(x, m) = (y, n)$, then $\sigma^k_A(x) = \sigma^l_A(y)$ and $m + l = n + k$.

There is a homomorphism $e_A : S^e_A \rightarrow \mathbb{Z}$ such that $e_A(\eta) = l - k$. An eventual conjugacy $h : X_A \rightarrow X_B$ induces an isomorphism $\psi : S^e_A \rightarrow S^e_B$ such that if $\eta \in S^e_A$ and $\text{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), n)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$.

Conversely, if $\psi : S^e_A \rightarrow S^e_B$ is an isomorphism such that $e_A = e_B \circ \psi$, then there is an eventual conjugacy $h : X_A \rightarrow X_B$ such that if $\eta \in S^e_A$ and $\text{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), n)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$.

The gauge action

There is an action $\mathbb{T} \ni t \mapsto \lambda^A_t \in \text{Aut}(O_A)$ such that $\lambda^A_t(s_i) = ts_i$ for $i \in \{1, \ldots, N\}$.

We have that the action $\gamma_t$ induced by $\lambda^A_t$ on $C^*(G_A)$ by the isomorphism between $O_A$ and $C^*(G_A)$ satisfies $\gamma_t(f)(x, n, y) = t^nf(x, n, y)$ for $f \in C^*(G_A) \subseteq C_0(G_A)$ and $(x, n, y) \in G_A$. It follows that if there is an isomorphism $\phi : G_A \rightarrow G_B$ such that $c_A = c_B \circ \phi$, then there is a diagonal-preserving isomorphism between $O_A$ and $O_B$ that intertwines the actions $\lambda^A_t$ and $\lambda^B_t$.

Conversely, James Rout and I showed that if there is a diagonal-preserving isomorphism between $O_A$ and $O_B$ that intertwines the actions $\lambda^A_t$ and $\lambda^B_t$, then the isomorphism $\phi : G_A \rightarrow G_B$ we get from this isomorphism satisfies $c_A = c_B \circ \phi$ (this also follows from the results of C–Ruiz–Sims–Tomforde).
The $\mathbb{Z}$-grading of $RG_A$

The cocycle $c_A : G_A \to \mathbb{Z}$ induces a $\mathbb{Z}$-grading $\{(RG_A)_n\}_{n \in \mathbb{Z}}$ given by $(RG_A)_n = \{f \in C_c(G_A, R) : \text{supp}(f) \subseteq c_A^{-1}(n)\}$.

It follows that if there is an isomorphism $\phi : G_A \to G_B$ such that $c_A = c_B \circ \phi$, then there is a graded diagonal-preserving isomorphism between $RG_A$ and $RG_B$.

Conversely, Steinberg’s result shows that if there is a graded diagonal-preserving isomorphism between $RG_A$ and $RG_B$, then there is an isomorphism $\phi : G_A \to G_B$ such that $c_A = c_B \circ \phi$.

Notice that we do not need to assume that $R$ is reduced here.

Theorem 3

Let $A$ and $B$ be finite square $\{0, 1\}$-matrices with no zero rows and no zero columns, and let $R$ be an indecomposable commutative ring with unit. The following are equivalent.

1. The one-sided shift spaces $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are conjugate.
2. There is an isomorphism $\phi : G_A \to G_B$ such that $\phi \circ \epsilon_A = \epsilon_B \circ \phi$.
3. There is an isomorphism $\psi : S_A \to S_B$ such that $\psi(S_A^r) = S_B^r$ and $\psi \circ r_A = r_B \circ \psi$.
4. The Cuntz–Krieger algebras $O_A$ and $O_B$ are isomorphic by a diagonal-preserving isomorphism that intertwines the positive maps $\tau_A$ and $\tau_B$.
5. The Steinberg algebras $RG_A$ and $RG_B$ are isomorphic by a diagonal-preserving isomorphism that intertwines $K_A$ and $K_B$. 
The map $\epsilon_A$

The map $\epsilon_A : G_A \to G_A$ is defined by
$\epsilon_A(x, n, y) = (\sigma_A(x), n, \sigma_A(y))$. It is a continuous groupoid homomorphism.

It is easy to see that if $h : X_A \to X_B$ is a conjugacy, then there is an isomorphism $\phi : G_A \to G_B$ such that $\phi(x, n, y) = (h(x), n, h(y))$ and $\phi \circ \epsilon_A = \epsilon_B \circ \phi$.

Conversely, if $\phi : G_A \to G_B$ is an isomorphism such that $\phi \circ \epsilon_A = \epsilon_B \circ \phi$, then the restriction of $\phi$ to $G_A^{(0)}$ induces a conjugacy $h : X_A \to X_B$.

The inverse semigroup $S^r_A$ and the homomorphism $r_A$

The inverse semigroup $S^r_A$ is the subsemigroup of $S_A$ consisting of homeomorphisms $\eta : \text{dom}(\eta) \to \text{ran}(\eta)$ where $\text{dom}(\eta) = U \times \mathbb{Z}$ and $\text{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets $U, V$ of $X_A$ such that the restrictions of $\sigma_A$ to each of $U$ and $V$ are injective.

The map $r_A : S^r_A \to S_A$ is defined by $r_A(\eta)(\sigma_A(x), n) = (\sigma_A(y), m)$ for $(x, n) \in \text{dom}(\eta)$ where $(y, m) = \eta(x, n)$.

A conjugacy $h : X_A \to X_B$ induces an isomorphism $\psi : S_A \to S_B$ such that if $\eta \in S_A$ and $\text{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), n)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$. This isomorphism $\psi : S_A \to S_B$ satisfies $\psi(S^r_A) = S^r_B$ and $\psi \circ r_A = r_B \circ \psi$. 
The inverse semigroup $S^r_A$ and the homomorphism $r_A$

Conversely, if $\psi : S_A \to S_B$ is an isomorphism such that $\psi(S^r_A) = S^r_B$ and $\psi \circ r_A = r_B \circ \psi$, then there is a conjugacy $h : X_A \to X_B$ such that if $\eta \in S_A$ and $\text{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), n)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$.

The positive map $\tau_A$

The map $\tau_A : \mathcal{O}_A \to \mathcal{O}_A$ is defined by $\tau_A(y) = \sum_{i,j=1}^N s_i y s_j^*$. It is a completely positive map.

If we identify $\mathcal{O}_A$ with $C^*(G_A) \subseteq C_0(G_A)$, then $\tau_A(f)(x, n, y) = f(\epsilon_A(x, n, y))$ for $f \in C^*(G_A) \subseteq C_0(G_A)$ and $(x, n, y) \in G_A$.

It follows that an isomorphism $\phi : G_A \to G_B$ such that $\phi \circ \epsilon_A = \epsilon_B \circ \phi$ induces a diagonal-preserving isomorphism between $\mathcal{O}_A$ and $\mathcal{O}_B$ that intertwines $\tau_A$ and $\tau_B$.

Conversely, if $\pi : \mathcal{O}_A \to \mathcal{O}_B$ is a diagonal-preserving isomorphism such that $\pi \circ \tau_A = \tau_B \circ \pi$, then $\pi(\mathcal{F}_A) = \mathcal{F}_B$ and $\pi(f \circ \sigma_A) = \phi(f) \circ \sigma_B$ for $f \in \mathcal{D}_A = C(X_A)$. It follows that $X_A$ and $X_B$ are conjugate.
The map $\kappa_A$

The map $\kappa_A : RG_A \rightarrow RG_A$ is defined by

$$\kappa_A(f)(x, n, y) = f(\epsilon_A(x, n, y)) \text{ for } f \in RG_A = C_c(G_A) \text{ and } (x, n, y) \in G_A.$$ 

It follows that an isomorphism $\phi : G_A \rightarrow G_B$ such that $\phi \circ \epsilon_A = \epsilon_B \circ \phi$ induces a diagonal-preserving isomorphism between $RG_A$ and $RG_B$ that intertwines $\kappa_A$ and $\kappa_B$. Conversely, if $\pi : RG_A \rightarrow RG_B$ is a diagonal-preserving isomorphism such that $\pi \circ \kappa_A = \kappa_B \circ \pi$, then

$$\pi(Rc_A^{-1}(0)) = Rc_B^{-1}(0) \text{ and } \pi(f \circ \sigma_A) = \phi(f) \circ \sigma_B$$

for $f \in RG_A^{(0)} = C(X_A, R)$, and it follows that $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are conjugate.

Theorem 4

Let $A$ and $B$ be finite square $\{0, 1\}$-matrices with no zero rows and no zero columns, and let $R$ be an indecomposable reduced commutative ring with unit. The following are equivalent.

1. The two-sided shift spaces $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow-equivalent.
2. The groupoids $G_A \times R$ and $G_B \times R$ are isomorphic.
3. The inverse semigroups $\tilde{S}_A$ and $\tilde{S}_B$ are isomorphic.
4. The stabilised Cuntz–Krieger algebras $O_A \otimes K$ and $O_B \otimes K$ are isomorphic by a diagonal-preserving isomorphism.
5. The algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a diagonal-preserving isomorphism.
Flow equivalence

The two-sided shift spaces $(\widebar{X}_A, \overline{\sigma}_A)$ consists of the totally disconnected compact Hausdorff space

$$\widebar{X}_A := \{(x_n)_{n \in \mathbb{Z}} \in \{1, 2, \ldots, N\}^\mathbb{Z} : A_{x_n, x_{n+1}} = 1 \text{ for all } n\}$$

and the homeomorphism $\overline{\sigma}_A : \widebar{X}_A \rightarrow \widebar{X}_A$ given by

$$(\overline{\sigma}_A((x_n)_{n \in \mathbb{Z}}))_m = x_{m+1}.$$ 

A discrete cross section of $\widebar{X}_A$ is a pair $(\mathcal{X}, \overline{\sigma})$ consisting of a closed subset $\mathcal{X} \subseteq \widebar{X}_A$ and a homeomorphism $\overline{\sigma} : \mathcal{X} \rightarrow \mathcal{X}$ such that there is a continuous function $r : \mathcal{X} \rightarrow \mathbb{N}$ such that $\overline{\sigma}(x) = \overline{\sigma}^{r(x)}(x)$ and $\overline{\sigma}_A(x) \notin \mathcal{X}$ for all $x \in \mathcal{X}$ and all $j \in \{1, 2, \ldots, r(x) - 1\}$, and $\widebar{X}_A = \{\overline{\sigma}_A^n(x) : x \in \mathcal{X}, \ n \in \mathbb{N}\}$.

Flow equivalence

It follows from a theorem of Parry and Sullivan that $(\widebar{X}_A, \overline{\sigma}_A)$ and $(\widebar{X}_B, \overline{\sigma}_B)$ are flow-equivalent if and only if there are discrete cross sections $(\mathcal{X}_1, \overline{\sigma}_1)$ and $(\mathcal{X}_2, \overline{\sigma}_2)$ of $(\widebar{X}_A, \overline{\sigma}_A)$ and $(\widebar{X}_B, \overline{\sigma}_B)$ respectively, such that $(\mathcal{X}_1, \overline{\sigma}_1)$ and $(\mathcal{X}_2, \overline{\sigma}_2)$ are conjugate.
The groupoid $\mathcal{R}$

We let $\mathcal{R}$ denote the discrete semigroup $\mathbb{N} \times \mathbb{N}$ where $(m, n)^{-1} = (n, m)$ and $(m, n)(n, p) = (m, p)$.

$G_A \times \mathcal{R}$ is an amenable, second countable, étale, and locally compact Hausdorff groupoid.

It follows from a result by C–Ruiz–Sims that $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic if and only if $G_A$ and $G_B$ are groupoid equivalent, if and only if $G_A$ and $G_B$ are Kakutani equivalent.

Using this, C–Eilers–Ortega-Restorff showed that if $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic, then $(\mathcal{X}_A, \mathcal{A}_A)$ and $(\mathcal{X}_B, \mathcal{A}_B)$ are flow-equivalent.

Conversely, by using the result of Parry and Sullivan, it is not difficult to show that if $(\mathcal{X}_A, \mathcal{A}_A)$ and $(\mathcal{X}_B, \mathcal{A}_B)$ are flow-equivalent, then $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic.

The inverse semigroup $\tilde{S}_A$

Let $\tilde{\sigma}_A : X_A \times \mathbb{N} \to X_A \times \mathbb{N}$ be defined by

$$
\tilde{\sigma}_A(x, n) = \begin{cases} (x, n - 1) & \text{if } n \neq 0, \\ (\sigma_A(x), 0) & \text{if } n = 0. \end{cases}
$$

Then $\tilde{\sigma}_A$ is a local homeomorphism.

The inverse semigroup $\tilde{S}_A$ consists of homeomorphisms $\eta : \text{dom}(\eta) \to \text{ran}(\eta)$ where $\text{dom}(\eta) = U \times \mathbb{Z}$ and $\text{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets $U, V$ of $X_A \times \mathbb{N}$, and there are continuous functions $k : U \to \mathbb{N}$ and $l : V \to \mathbb{N}$ such that if $\eta(x, m) = (y, n)$, then $\tilde{\sigma}_A^{k(x)}(x) = \tilde{\sigma}_A^{l(y)}(y)$ and $m + l(y) = n + k(x)$.

We have that the inverse semigroups $\tilde{S}_A$ and $\tilde{S}_B$ are isomorphic if and only if $(X_A \times \mathbb{N}, \tilde{S}_A)$ and $(X_B \times \mathbb{N}, \tilde{S}_B)$ are continuously orbit equivalent, if and only if the groupoids $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic.
Stabilised Cuntz–Krieger algebras

The stabilised Cuntz–Krieger algebra $O_A \otimes K$ is isomorphic to $C^*(G_A \times \mathcal{R})$ by an isomorphism that maps $D_A \otimes C$, where $C$ is the canonical maximal abelian subalgebra of $K$, onto $C_0(G_A \times \mathcal{R})^{(0)}$. It follows that if $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic, then the stabilised Cuntz–Krieger algebras $O_A \otimes K$ and $O_B \otimes K$ are isomorphic by a diagonal-preserving isomorphism.

Conversely, if the stabilised Cuntz–Krieger algebras $O_A \otimes K$ and $O_B \otimes K$ are isomorphic by a diagonal-preserving isomorphism, then it follows from the result of Brownlowe-C–Whittaker or the result of C–Ruiz-Sims-Tomforde that $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic. Cuntz–Krieger (for irreducible matrices satisfying condition (I)) and Cuntz (for matrices for which the matrices and their transpose satisfy condition (I)) showed that if $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent, then the stabilised Cuntz–Krieger algebras $O_A \otimes K$ and $O_B \otimes K$ are isomorphic by a diagonal-preserving isomorphism. Matsumoto–Matui showed, using results of Boyle–Handleman, Franks, and Huang, that for irreducible matrices diagonal-preserving isomorphism of $O_A \otimes K$ and $O_B \otimes K$ implies that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.
The algebra $RG_A \otimes M_\infty(R)$

The algebra $RG_A \otimes M_\infty(R)$ is isomorphic to $R(G_A \times \mathcal{R})$ by an isomorphism that maps $RG_A^{(0)} \otimes D_\infty(R)$ onto $R(G_A \times \mathcal{R})^{(0)}$. It follows that if $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic, then the algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a diagonal-preserving isomorphism. Conversely, if the algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a diagonal-preserving isomorphism, then it follows from the result of Steinberg that $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic.

Theorem 5

Let $A$ and $B$ be finite square $\{0, 1\}$-matrices with no zero rows and no zero columns, and let $R$ be an indecomposable commutative ring with unit. The following are equivalent.

1. The two-sided shift spaces $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are conjugate.

2. There is an isomorphism $\phi : G_A \times \mathcal{R} \to G_B \times \mathcal{R}$ such that $\bar{c}_A = \bar{c}_B \circ \phi$.

3. There is an isomorphism $\psi : \bar{S}_A^e \to \bar{S}_B^e$ such that $\bar{e}_A = \bar{e}_B \circ \psi$.

4. The stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism that intertwines the actions $\lambda^A_t \otimes \text{id}_\mathcal{K}$ and $\lambda^B_t \otimes \text{id}_\mathcal{K}$.

5. The algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a graded diagonal-preserving isomorphism.
Conjugacy of the two-sided shift spaces $(\widetilde{X}_A, \widetilde{\sigma}_A)$ and $(\widetilde{X}_B, \widetilde{\sigma}_B)$ and graded isomorphism of $G_A \times \mathbb{R}$ and $G_B \times \mathbb{R}$

Let $\tilde{c}_A : G_A \times \mathbb{R} \to \mathbb{Z}$ be the cocycle defined by
$\tilde{c}_A(\eta, (m, n)) = c_A(\eta)$ for $\eta \in G_A$.

James Rout and I showed that the two-sided shift spaces $(\widetilde{X}_A, \widetilde{\sigma}_A)$ and $(\widetilde{X}_B, \widetilde{\sigma}_B)$ are conjugate if and only if there is an isomorphism $\phi : G_A \times \mathbb{R} \to G_B \times \mathbb{R}$ such that $\tilde{c}_A = \tilde{c}_B \circ \phi$.

The inverse semigroup $\tilde{S}_e^A$ and the homomorphism $\tilde{e}_A : \tilde{S}_e^A \to \mathbb{Z}$

The inverse semigroup $\tilde{S}_e^A$ is the subsemigroup of $S_A$ consisting of homeomorphisms $\eta : \text{dom}(\eta) \to \text{ran}(\eta)$ where $\text{dom}(\eta) = U \times \mathbb{Z}$ and $\text{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets $U, V$ of $X_A \times \mathbb{N}$, and there are $k, l \in \mathbb{N}$ such that if $\eta(x, m) = (y, n)$, then $\tilde{\sigma}_A^k(x) = \tilde{\sigma}_A^l(y)$ and $m + l = n + k$.

There is a homomorphism $\tilde{e}_A : \tilde{S}_e^A \to \mathbb{Z}$ such that $\tilde{e}_A(\eta) = l - k$.

Using techniques similar to the once James Rout and I used, one can show that there is an isomorphism $\psi : \tilde{S}_e^A \to \tilde{S}_e^B$ such that $\tilde{e}_A = \tilde{e}_B \circ \psi$ if and only if $(\widetilde{X}_A, \widetilde{\sigma}_A)$ and $(\widetilde{X}_B, \widetilde{\sigma}_B)$ are conjugate.
Graded isomorphisms of the algebras

If there is an isomorphism $\phi : G_A \times R \to G_B \times R$ such that $\check{c}_A = \check{c}_B \circ \phi$, then the stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism that intertwines the actions $\lambda_t^A \otimes \text{id}_\mathcal{K}$ and $\lambda_t^B \otimes \text{id}_\mathcal{K}$, and the algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a graded diagonal-preserving isomorphism. Conversely, James Rout and I showed that if the stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism that intertwines the actions $\lambda_t^A \otimes \text{id}_\mathcal{K}$ and $\lambda_t^B \otimes \text{id}_\mathcal{K}$, then the isomorphism $\phi : G_A \times R \to G_B \times R$ we get from the Brownlowe–C–Whittaker results, satisfies $\check{c}_A = \check{c}_B \circ \phi$ (this also follows from the result of C–Ruiz–Sims–Tomforde).

Similarly, it follows from Steinberg’s result that if the algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a graded diagonal-preserving isomorphism, then there is an isomorphism $\phi : G_A \times R \to G_B \times R$ such that $\check{c}_A = \check{c}_B \circ \phi$. 