Shifts of finite type, Cuntz–Krieger algebras and their algebraic analogues, groupoids, and inverse semigroups

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Theorem 1

Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns, and let R be an indecomposable reduced commutative ring with unit. The following are equivalent.

1. The one-sided shift spaces (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.
2. The groupoids G_A and G_B are isomorphic as topological groupoids.
3. The inverse semigroups S_A and S_B are isomorphic.
4. The Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic by a diagonal-preserving isomorphism.
5. The Steinberg algebras RG_A and RG_B are isomorphic by a diagonal-preserving isomorphism.

One-sided shifts of finite type

Let $A \in M_N(\{0,1\})$ be a matrix with no zero rows and no zero columns. The space

$$X_A := \left\{(x_n)_{n\in\mathbb{N}} \in \{1,2,\dots,N\}^{\mathbb{N}} : A_{x_n,x_{n+1}} = 1 \text{ for all } n\right\}$$

is a totally disconnected compact Hausdorff space.
The map $\sigma_A : X_A \to X_A$

$$x_0x_1x_2\cdots \mapsto x_1x_2\dots$$

is a local homeomorphism. The pair (X_A, σ_A) is a shift of finite type.
The *orbit* of $x \in X_A$ is

$$\bigcup_{m,n\in\mathbb{N}} \sigma_A^m(\sigma_A^{-n}(\{x\})) = \left\{y \in X_A : \sigma_A^k(x) = \sigma_A^l(y) \text{ for some } k,l \in \mathbb{N}\right\}.$$

Continuous orbit equivalence

Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns. If $h : X_A \to X_B$ is a homeomorphism that maps orbits to orbits, then for each $x \in X_A$ there exist nonnegative integers k_x, l_x such that

$$\sigma_B^{k_x}(h(\sigma_A(x))) = \sigma_B^{l_x}(h(x)).$$

We say that h is a *continuous orbit equivalence* and that (X_A, σ_A) and (X_B, σ_B) are *continuously orbit equivalent* if there are continous functions $k, l : X_A \to \mathbb{N}$ and $k', l' : X_B \to \mathbb{N}$ such that

$$\sigma_B^{k(x)}(h(\sigma_A(x))) = \sigma_B^{l(x)}(h(x)) \quad \sigma_A^{k'(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l'(y)}(h^{-1}(y))$$

for all $x \in X_A$ and all $y \in X_B$.
This notion was, in this setting, introduced by Matsumoto.

The groupoid G_A

The *Deaconu-Renault groupoid* associated to the one-sided shift of finite type (X_A, σ_A) is

$$G_A := \{(x, n, y) \in X_A \times \mathbb{Z} \times X_A \mid \exists k, l \in \mathbb{N},\ n = k - l \colon \sigma_A^k(x) = \sigma_A^l(y)\}$$

with unit space $G_A^{(0)} = \{(x, 0, x) \in G_A \mid x \in X_A\}$. The range map is $r(x, n, y) = (x, 0, x)$ and the source map is $s(x, n, y) = (y, 0, y)$. The product $(x, n, y)(x', n', y')$ is well-defined if and only if $y = x'$ in which case it equals $(x, n + n', y')$, while inversion is given by $(x, n, y)^{-1} = (y, -n, x)$. The topology on G_A has a basis consisting of sets of the form

$$Z(U, k, l, V) := \{(x, k - l, y) \in G_A \mid x \in U, y \in V\},$$

where $k, l \in \mathbb{N}$ and $U, V \subseteq X_A$ are open such that $\sigma_A^k|_U$ and $\sigma_A^l|_V$ are injective and $\sigma_A^k(U) = \sigma_A^l(V)$. With this, G_A is an amenable, second countable, étale, and locally compact Hausdorff groupoid.

Continuous orbit equivalence and groupoid isomorphism

If $h : X_A \to X_B$ is a continuous orbit equivalence, then there exists a function $m : G_A \to \mathbb{Z}$ such that $(x, n, y) \mapsto (h(x), m(x, n, y), h(y))$ is an isomorphism between G_A and G_B (one have to take a little care if there are isolated periodic points).
Conversely, if $\phi : G_A \to G_B$ is an isomorphism, then ϕ restricts to a homeomorphism between $G_A^{(0)}$ and $G_B^{(0)}$ and thus induces a homeomorphism $h : X_A \to X_B$ such that $\phi(x, 0, x) = (h(x), 0, h(x))$ for $x \in X_A$, and such that h is a continuous orbit equivalence.

The inverse semigroup S_A

The inverse semigroup S_A consists of homeomorphisms $\eta : \mathrm{dom}(\eta) \to \mathrm{ran}(\eta)$ where $\mathrm{dom}(\eta) = U \times \mathbb{Z}$ and $\mathrm{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets U, V of X_A, and there are continuous functions $k : U \to \mathbb{N}$ and $l : V \to \mathbb{N}$ such that if $\eta(x, m) = (y, n)$, then $\sigma_A^{k(x)}(x) = \sigma_A^{l(y)}(y)$ and $m + l(y) = n + k(x)$.
A continuous orbit equivalence $h : X_A \to X_B$ induces an isomorphism $\psi : S_A \to S_B$ such that if $\eta \in S_A$ and $\mathrm{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), m + k)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$ and k is some integer that depends on x and η.
The set of characters of the idempotent semi-lattice $\mathcal{E}(S_A)$ of S_A is homeomorphic to X_A, and the groupoid of germs of the action of S_A on the set of characters of $\mathcal{E}(S_A)$ is isomorphic to G_A. It follows that an isomorphism between S_A and S_B induces an isomorphism between G_A and G_B.

Cuntz–Krieger algebras

The *Cuntz–Krieger algebra* $\mathcal{O}_A$ of A is the universal unital C^*-algebra generated by partial isometries $s_1 \ldots, s_N$ subject to the conditions

$$s_i^* s_j = 0 \ (i \neq j), \qquad {s_i}^* s_i = \sum_{j=1}^{N} A_{i,j} s_j {s_j}^*$$

for every $i = 1, \ldots, N$.
If $\alpha = \alpha_1 \ldots \alpha_n$ is a word of elements from $\{1, \ldots, N\}$, then we let $s_\alpha := s_{\alpha_1} \ldots s_{\alpha_n}$, and we let

$$\mathcal{D}_A := \overline{\mathrm{span}}\{s_\alpha s_\alpha^* : \alpha \text{ is a finite word}\}.$$

Then $\mathcal{D}_A$ is isomorphic to $C(X_A)$.

Cuntz–Krieger algebras

There is an isomorphism from $C^*(G_A)$ to $\mathcal{O}_A$ that maps $C(G_A^{(0)})$ onto $\mathcal{D}_A$. It follows that if G_A and G_B are isomorphic, then there is an isomorphism $\phi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\phi(\mathcal{D}_A) = \mathcal{D}_B$. Conversely, if there is an isomorphism $\phi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\phi(\mathcal{D}_A) = \mathcal{D}_B$, then G_A and G_B are isomorphic. If A and B both satisfy the condition introduced by Cuntz and Krieger called condition (I), then this follows from a result by Renault which says that two second countable locally compact topological principal étale Hausdorff groupoids are isomorphic if and only if there is a diagonal-preserving isomorphism between their reduced C^*-algebras. The general case follows from a result by Brownlowe–C–Whittaker that says the groupoids of two countable directed graphs are isomorphic if and only if there is a diagonal-preserving isomorphism between their C^*-algebras.

Reconstruction of groupoids

This has recently been generalised by C–Ruiz-Sims-Tomforde that show that two second countable locally compact étale Hausdorff groupoids for which the interior of the isotropi is abelian and torsion-free, are isomorphic if and only if there is a diagonal-preserving isomorphism between their reduced C^*-algebras.

The equivalence of (1) and (4) was also proven by Matsumoto for irreducible matrices satisfying condition (I) without the use of groupoids.

Steinberg algebras

The Steinberg algebra RG_A of G_A is $C_c(G_A, R)$ equipped with a convolution product defined similar to how the product of $C^*(G_A)$ is defined. If $R = \mathbb{C}$, then RG_A is a dense subalgebra of $C^*(G_A)$.

It follows that if G_A and G_B are isomorphic, then there is an isomorphism from RG_A to RG_B that maps $RG_A^{(0)}$ onto $RG_B^{(0)}$.

The converse follows from a recent result of Steinberg which says that if R is an indecomposable commutative ring with unit, and G_1 and G_2 are Hausdoff ample groupoids such that G_1 has a dense set of objects x such that the group algebra over R of the isotropy group at x of the interior of the isotropy bundle of G_1 has no non-trivial units, then G_1 and G_2 are isomorphic if and only if their is a diagonal-preserving isomorphism between RG_1 and RG_2.

Theorem 2

Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns, and let R be an indecomposable commutative ring with unit. The following are equivalent.

1. The one-sided shift spaces (X_A, σ_A) and (X_B, σ_B) are eventually conjugate.
2. There is an isomorphism $\phi : G_A \to G_B$ such that $c_A = c_B \circ \phi$.
3. There is an isomorphism $\psi : S_A^e \to S_B^e$ such that $e_A = e_B \circ \psi$.
4. The Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic by a diagonal-preserving isomorphism that intertwines the gauge actions λ_t^A and λ_t^B.
5. The Steinberg algebras RG_A and RG_B are isomorphic by a graded diagonal-preserving isomorphism.

Eventually conjugacy

Let $h: X_A \to X_B$ be a homeomorphism. We say that h is an *eventual conjugacy* and that (X_A, σ_A) and (X_B, σ_B) are *eventually conjugate* if there is an $k \in \mathbb{N}$ such that

$$\sigma_B^k(h(\sigma_A(x))) = \sigma_B^{k+1}(h(x)) \quad \sigma_A^k(h^{-1}(\sigma_B(y))) = \sigma_A^{k+1}(h^{-1}(y))$$

for all $x \in X_A$ and all $y \in X_B$. This is a notion introduced by Matsumoto.

The cocycle c_A

The function $c_A : G_A \to \mathbb{Z}$ is defined by $c_A(x, n, y) = n$. This is a continous cocycle in the sense $c_A((x, n, y)^{-1}) = -c_A(x, n, y)$ and $c_A((x, n, y)(y, m, z)) = c_A(x, n + m, z)$.
If $h: X_A \to X_B$ is an eventual conjugacy, then $(x, n, y) \mapsto (h(x), n, h(y))$ is an isomorphism between G_A and G_B such that $c_A = c_B \circ \phi$.
Conversely, if $\phi : G_A \to G_B$ is an isomorphism such that $c_A = c_B \circ \phi$, then ϕ restricts to a homeomorphism between $G_A^{(0)}$ and $G_B^{(0)}$ and thus induces a homeomorphism $h : X_A \to X_B$ such that $\phi(x, 0, x) = (h(x), 0, h(x))$ for $x \in X_A$, and such that h is an eventual conjugacy.

The inverse semigroup S_A^e and the homomorphism $e_A : S_A^e \to \mathbb{Z}$

The inverse semigroup S_A^e is the subsemigroup of S_A consisting of homeomorphisms $\eta : \mathrm{dom}(\eta) \to \mathrm{ran}(\eta)$ where $\mathrm{dom}(\eta) = U \times \mathbb{Z}$ and $\mathrm{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets U, V of X_A, and there are $k, l \in \mathbb{N}$ such that if $\eta(x, m) = (y, n)$, then $\sigma_A^k(x) = \sigma_A^l(y)$ and $m + l = n + k$.
There is a homomorphism $e_A : S_A^e \to \mathbb{Z}$ such that $e_A(\eta) = l - k$.
An eventual conjugacy $h : X_A \to X_B$ induces an isomorphism $\psi : S_A^e \to S_B^e$ such that if $\eta \in S_A^e$ and $\mathrm{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), n)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$.
Conversely, if $\psi : S_A^e \to S_B^e$ is an isomorphism such that $e_A = e_B \circ \psi$, then there is an eventual conjugacy $h : X_A \to X_B$ such that if $\eta \in S_A^e$ and $\mathrm{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), n)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$.

The gauge action

There is an action $\mathbb{T} \ni t \mapsto \lambda_t^A \in \mathrm{Aut}(\mathcal{O}_A)$ such that $\lambda_t^A(s_i) = ts_i$ for $i \in \{1, \ldots, N\}$.
We have that the action γ_t induced by λ_t^A on $C^*(G_A)$ by the isomorphism between $\mathcal{O}_A$ and $C^*(G_A)$ satisfies $\gamma_t(f)(x, n, y) = t^n f(x, n, y)$ for $f \in C^*(G_A) \subseteq C_0(G_A)$ and $(x, n, y) \in G_A$. It follows that if there is an isomorphism $\phi : G_A \to G_B$ such that $c_A = c_B \circ \phi$, then there is a diagonal-preserving isomorphism between $\mathcal{O}_A$ and $\mathcal{O}_B$ that intertwines the actions λ_t^A and λ_t^B.
Conversely, James Rout and I showed that if there is a diagonal-preserving isomorphism between $\mathcal{O}_A$ and $\mathcal{O}_B$ that intertwines the actions λ_t^A and λ_t^B, then the isomorphism $\phi : G_A \to G_B$ we get from this isomorphism satisfies $c_A = c_B \circ \phi$ (this also follows from the results of C–Ruiz–Sims–Tomforde).

The $\mathbb{Z}$-grading of RG_A

The cocycle $c_A : G_A \to \mathbb{Z}$ induces a $\mathbb{Z}$-grading $\{(RG_A)_n\}_{n\in\mathbb{Z}}$ given by $(RG_A)_n = \{f \in C_c(G_A, R) : \operatorname{supp}(f) \subseteq c_A^{-1}(n)\}$.
It follows that if there is an isomorphism $\phi : G_A \to G_B$ such that $c_A = c_B \circ \phi$, then there is a graded diagonal-preserving isomorphism between RG_A and RG_B.
Conversely, Steinberg's result shows that if there is a graded diagonal-preserving isomorphism between RG_A and RG_B, then there is an isomorphism $\phi : G_A \to G_B$ such that $c_A = c_B \circ \phi$.
Notice that we do not need to assume that R is reduced here.

Theorem 3

Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns, and let R be an indecomposable commutative ring with unit. The following are equivalent.

1. The one-sided shift spaces (X_A, σ_A) and (X_B, σ_B) are conjugate.
2. There is an isomorphism $\phi : G_A \to G_B$ such that $\phi \circ \epsilon_A = \epsilon_B \circ \phi$.
3. There is an isomorphism $\psi : S_A \to S_B$ such that $\psi(S_A^r) = S_B^r$ and $\psi \circ r_A = r_B \circ \psi$.
4. The Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic by a diagonal-preserving isomorphism that intertwines the positive maps τ_A and τ_B.
5. The Steinberg algebras RG_A and RG_B are isomorphic by a diagonal-preserving isomorphism that intertwines κ_A and κ_B.

The map ϵ_A

The map $\epsilon_A : G_A \to G_A$ is defined by $\epsilon_A(x, n, y) = (\sigma_A(x), n, \sigma_A(y))$. It is a continuous groupoid homomorphism.
It is easy to see that if $h : X_A \to X_B$ is a conjugacy, then there is an isomorphism $\phi : G_A \to G_B$ such that $\phi(x, n, y) = (h(x), n, h(y))$ and $\phi \circ \epsilon_A = \epsilon_B \circ \phi$.
Conversely, if $\phi : G_A \to G_B$ is an isomorphism such that $\phi \circ \epsilon_A = \epsilon_B \circ \phi$, then the restriction of ϕ to $G_A^{(0)}$ induces a conjugacy $h : X_A \to X_B$.

The inverse semigroup S_A^r and the homomorphism r_A

The inverse semigroup S_A^r is the subsemigroup of S_A consisting of homeomorphisms $\eta : \mathrm{dom}(\eta) \to \mathrm{ran}(\eta)$ where $\mathrm{dom}(\eta) = U \times \mathbb{Z}$ and $\mathrm{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets U, V of X_A such that the restrictions of σ_A to each of U and V are injective.
The map $r_A : S_A^r \to S_A$ is defined by $r_A(\eta)(\sigma_A(x), n) = (\sigma_A(y), m)$ for $(x, n) \in \mathrm{dom}(\eta)$ where $(y, m) = \eta(x, n)$.
A conjugacy $h : X_A \to X_B$ induces an isomorphism $\psi : S_A \to S_B$ such that if $\eta \in S_A$ and $\mathrm{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), n)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$. This isomorphism $\psi : S_A \to S_B$ satisfies $\psi(S_A^r) = S_B^r$ and $\psi \circ r_A = r_B \circ \psi$.

The inverse semigroup S^r_A and the homomorphism r_A

Conversely, if $\psi : S_A \to S_B$ is an isomorphism such that $\psi(S^r_A) = S^r_B$ and $\psi \circ r_A = r_B \circ \psi$, then there is a conjugacy $h : X_A \to X_B$ such that if $\eta \in S_A$ and $\mathrm{dom}(\eta) = U \times \mathbb{Z}$, then $\psi(\eta)(x, m) = (h(y), n)$ for $(x, m) \in h(U) \times \mathbb{Z}$ where $(y, n) = \eta(h^{-1}(x), m)$.

The positive map τ_A

The map $\tau_A : \mathcal{O}_A \to \mathcal{O}_A$ is defined by $\tau_A(y) = \sum_{i,j=1}^N s_i y s_j^*$. It is a completely positive map.
If we identify $\mathcal{O}_A$ with $C^*(G_A) \subseteq C_0(G_A)$, then $\tau_A(f)(x, n, y) = f(\epsilon_A(x, n, y))$ for $f \in C^*(G_A) \subseteq C_0(G_A)$ and $(x, n, y) \in G_A$.
It follows that an isomorphism $\phi : G_A \to G_B$ such that $\phi \circ \epsilon_A = \epsilon_B \circ \phi$ induces a diagonal-preserving isomorphism between $\mathcal{O}_A$ and $\mathcal{O}_B$ that intertwines τ_A and τ_B.
Conversely, if $\pi : \mathcal{O}_A \to \mathcal{O}_B$ is a diagonal-preserving isomorphism such that $\pi \circ \tau_A = \tau_B \circ \pi$, then $\pi(\mathcal{F}_A) = \mathcal{F}_B$ and $\pi(f \circ \sigma_A) = \phi(f) \circ \sigma_B$ for $f \in \mathcal{D}_A = C(X_A)$. It follows that X_A and X_B are conjugate.

The map κ_A

The map $\kappa_A : RG_A \to RG_A$ is defined by $\kappa_A(f)(x,n,y) = f(\epsilon_A(x,n,y))$ for $f \in RG_A = C_c(G_A)$ and $(x,n,y) \in G_A$.
It follows that an isomorphism $\phi : G_A \to G_B$ such that $\phi \circ \epsilon_A = \epsilon_B \circ \phi$ induces a diagonal-preserving isomorphism between RG_A and RG_B that intertwines κ_A and κ_B.
Conversely, if $\pi : RG_A \to RG_B$ is a diagonal-preserving isomorphism such that $\pi \circ \kappa_A = \kappa_B \circ \pi$, then $\pi(Rc_A^{-1}(0)) = Rc_B^{-1}(0)$ and $\pi(f \circ \sigma_A) = \phi(f) \circ \sigma_B$ for $f \in RG_A^{(0)} = C(X_A, R)$, and it follows that (X_A, σ_A) and (X_B, σ_B) are conjugate.

Theorem 4

Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns, and let R be an indecomposable reduced commutative ring with unit. The following are equivalent.

1. The two-sided shift spaces $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are flow-equivalent.
2. The groupoids $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic.
3. The inverse semigroups $\tilde{S}_A$ and $\tilde{S}_B$ are isomorphic.
4. The stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism.
5. The algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a diagonal-preserving isomorphism.

Flow equivalence

The two-sided shift spaces $(\overline{X}_A, \overline{\sigma}_A)$ consists of the totally disconnected compact Hausdorff space

$$\overline{X}_A := \left\{(x_n)_{n\in\mathbb{Z}} \in \{1, 2, \ldots, N\}^{\mathbb{Z}} : A_{x_n, x_{n+1}} = 1 \text{ for all } n\right\}$$

and the homeomorphism $\overline{\sigma}_A : \overline{X}_A \to \overline{X}_A$ given by $(\overline{\sigma}_A((x_n)_{n\in\mathbb{Z}}))_m = x_{m+1}$.
A *discrete cross section* of $\overline{X}_A$ is a pair $(\overline{X}, \overline{\sigma})$ consisting of a closed subset $\overline{X} \subseteq \overline{X}_A$ and a homeomorphism $\overline{\sigma} : \overline{X} \to \overline{X}$ such that there is a continuous function $r : \overline{X} \to \mathbb{N}$ such that $\overline{\sigma}(x) = \overline{\sigma}_A^{r(x)}(x)$ and $\overline{\sigma}_A^j(x) \notin \overline{X}$ for all $x \in \overline{X}$ and all $j \in \{1, 2, \ldots, r(x) - 1\}$, and $\overline{X}_A = \{\overline{\sigma}_A^n(x) : x \in \overline{X},\ n \in \mathbb{N}\}$.

Flow equivalence

It follows from a theorem of Parry and Sullivan that $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are flow-equivalent if and only if there are discrete cross sections $(\overline{X}_1, \overline{\sigma}_1)$ and $(\overline{X}_2, \overline{\sigma}_2)$ of $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ respectively, such that $(\overline{X}_1, \overline{\sigma}_1)$ and $(\overline{X}_2, \overline{\sigma}_2)$ are conjugate.

The groupoid $\mathcal{R}$

We let $\mathcal{R}$ denote the discrete semigroup $\mathbb{N} \times \mathbb{N}$ where $(m,n)^{-1} = (n,m)$ and $(m,n)(n,p) = (m,p)$.
$G_A \times \mathcal{R}$ is an amenable, second countable, étale, and locally compact Hausdorff groupoid.
It follows from a result by C–Ruiz–Sims that $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic if and only if G_A and G_B are groupoid equivalent, if and only if G_A and G_B are Kakutani equivalent.
Using this, C–Eilers–Ortega-Restorff showed that if $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic, then $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are flow-equivalent.
Conversely, by using the result of Parry and Sullivan, it is not difficult to show that if $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are flow-equivalent, then $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic.

The inverse semigroup $\tilde{S}_A$

Let $\tilde{\sigma}_A : X_A \times \mathbb{N} \to X_A \times \mathbb{N}$ be defined by

$$\tilde{\sigma}_A(x,n) = \begin{cases} (x, n-1) & \text{if } n \neq 0, \\ (\sigma_A(x), 0) & \text{if } n = 0. \end{cases}$$

Then $\tilde{\sigma}_A$ is a local homeomorphism.
The inverse semigroup $\tilde{S}_A$ consists of homeomorphisms $\eta : \mathrm{dom}(\eta) \to \mathrm{ran}(\eta)$ where $\mathrm{dom}(\eta) = U \times \mathbb{Z}$ and $\mathrm{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets U, V of $X_A \times \mathbb{N}$, and there are continuous functions $k : U \to \mathbb{N}$ and $l : V \to \mathbb{N}$ such that if $\eta(x,m) = (y,n)$, then $\tilde{\sigma}_A^{k(x)}(x) = \tilde{\sigma}_A^{l(y)}(y)$ and $m + l(y) = n + k(x)$.
We have that the inverse semigroups $\tilde{S}_A$ and $\tilde{S}_B$ are isomorphic if and only if $(X_A \times \mathbb{N}, \tilde{S}_A)$ and $(X_B \times \mathbb{N}, \tilde{S}_B)$ are continuously orbit equivalent, if and only if the groupoids $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic.

Stabilised Cuntz–Krieger algebras

The stabilised Cuntz–Krieger algebra $\mathcal{O}_A \otimes \mathcal{K}$ is isomorphic to $C^*(G_A \times \mathcal{R})$ by an isomorphism that maps $\mathcal{D}_A \otimes \mathcal{C}$, where $\mathcal{C}$ is the canonical maximal abelian subalgebra of $\mathcal{K}$, onto $C_0((G_A \times \mathcal{R})^{(0)})$. It follows that if $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic, then the stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism.

Stabilised Cuntz–Krieger algebras

Conversely, if the stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism, then it follows from the result of Brownlowe-C–Whittaker or the result of C–Ruiz–Sims-Tomforde that $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic.
Cuntz–Krieger (for irreducible matrices satisfying condition (I)) and Cuntz (for matrices for which the matrices and their transpose satisfy condition (I)) showed that if $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are flow equivalent, then the stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism. Matsumoto–Matui showed, using results of Boyle–Handleman, Franks, and Huang, that for irreducible matrices diagonal-preserving isomorphism of $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ implies that $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are flow equivalent,

The algebra $RG_A \otimes M_\infty(R)$

The algebra $RG_A \otimes M_\infty(R)$ is isomorphic to $R(G_A \times \mathcal{R})$ by an isomorphism that maps $RG_A^{(0)} \otimes D_\infty(R)$ onto $R(G_A \times \mathcal{R})^{(0)}$. It follows that if $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic, then the algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a diagonal-preserving isomorphism.
Conversely, if the algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a diagonal-preserving isomorphism, then it follows from the result of Steinberg that $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$ are isomorphic.

Theorem 5

Let A and B be finite square $\{0, 1\}$-matrices with no zero rows and no zero columns, and let R be an indecomposable commutative ring with unit. The following are equivalent.

1. The two-sided shift spaces $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are conjugate.
2. There is an isomorphism $\phi : G_A \times \mathcal{R} \to G_B \times \mathcal{R}$ such that $\tilde{c}_A = \tilde{c}_B \circ \phi$.
3. There is an isomorphism $\psi : \tilde{S}^e_A \to \tilde{S}^e_B$ such that $\tilde{e}_A = \tilde{e}_B \circ \psi$.
4. The stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism that intertwines the actions $\lambda^A_t \otimes \mathrm{id}_\mathcal{K}$ and $\lambda^B_t \otimes \mathrm{id}_\mathcal{K}$.
5. The algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a graded diagonal-preserving isomorphism.

Conjugacy of the two-sided shift spaces $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ and graded isomorphism of $G_A \times \mathcal{R}$ and $G_B \times \mathcal{R}$

Let $\tilde{c}_A : G_A \times \mathcal{R} \to \mathbb{Z}$ be the cocycle defined by $\tilde{c}_A(\eta, (m, n)) = c_A(\eta)$ for $\eta \in G_A$.
James Rout and I showed that the two-sided shift spaces $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are conjugate if and only if there is an isomorphism $\phi : G_A \times \mathcal{R} \to G_B \times \mathcal{R}$ such that $\tilde{c}_A = \tilde{c}_B \circ \phi$.

The inverse semigroup $\tilde{S}^e_A$ and the homomorphism $\tilde{e}_A : \tilde{S}^e_A \to \mathbb{Z}$

The inverse semigroup $\tilde{S}^e_A$ is the subsemigroup of S_A consisting of homeomorphisms $\eta : \mathrm{dom}(\eta) \to \mathrm{ran}(\eta)$ where $\mathrm{dom}(\eta) = U \times \mathbb{Z}$ and $\mathrm{ran}(\eta) = V \times \mathbb{Z}$ for compact open subsets U, V of $X_A \times \mathbb{N}$, and there are $k, l \in \mathbb{N}$ such that if $\eta(x, m) = (y, n)$, then $\tilde{\sigma}^k_A(x) = \tilde{\sigma}^l_A(y)$ and $m + l = n + k$.
There is a homomorphism $\tilde{e}_A : \tilde{S}^e_A \to \mathbb{Z}$ such that $\tilde{e}_A(\eta) = l - k$.
Using techniques similar to the once James Rout and I used, one can show that there is an isomorphism $\psi : \tilde{S}^e_A \to \tilde{S}^e_B$ such that $\tilde{e}_A = \tilde{e}_B \circ \psi$ if and only if $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are conjugate.

Graded isomorphisms of the algebras

If there is an isomorphism $\phi : G_A \times \mathcal{R} \to G_B \times \mathcal{R}$ such that $\tilde{c}_A = \tilde{c}_B \circ \phi$, then the stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism that intertwines the actions $\lambda_t^A \otimes \mathrm{id}_{\mathcal{K}}$ and $\lambda_t^B \otimes \mathrm{id}_{\mathcal{K}}$, and the algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a graded diagonal-preserving isomorphism. Conversely, James Rout and I showed that if the stabilised Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}$ and $\mathcal{O}_B \otimes \mathcal{K}$ are isomorphic by a diagonal-preserving isomorphism that intertwines the actions $\lambda_t^A \otimes \mathrm{id}_{\mathcal{K}}$ and $\lambda_t^B \otimes \mathrm{id}_{\mathcal{K}}$, then the isomorphism $\phi : G_A \times \mathcal{R} \to G_B \times \mathcal{R}$ we get from the Brownlowe–C–Whittaker results, satisfies $\tilde{c}_A = \tilde{c}_B \circ \phi$ (this also follows from the result of C–Ruiz–Sims–Tomforde).

Graded isomorphisms of the algebras

Similarly, it follows from Steinberg's result that if the algebras $RG_A \otimes M_\infty(R)$ and $RG_B \otimes M_\infty(R)$ are isomorphic by a graded diagonal-preserving isomorphism, then there is an isomorphism $\phi : G_A \times \mathcal{R} \to G_B \times \mathcal{R}$ such that $\tilde{c}_A = \tilde{c}_B \circ \phi$.