Generalized Stone Duality

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Non-Commutative Stone Dualities

- In recent years various non-commutative Stone dualities considered by Kudryavtseva, Lawson, Lenz, Resende, etc.
- These extend one of two commutative dualities –
  1. Classic Stone duality (Stone 1936):

     $$\text{Stone spaces} \quad \leftrightarrow \quad \text{Boolean algebras.}$$

     (Stone space = 0-dimensional compact Hausdorff space)
  2. Point-free topology (late 50's-):

     $$\text{sober spaces} \quad \leftrightarrow \quad \text{spatial frames.}$$
- Our goal: find a happy medium, a more faithful extension of classic Stone duality to non-0-dimensional spaces/groupoids.
- Somewhat similar to Shirota (1952)/De Vries (1962) duality:

$$\text{compact Hausdorff spaces} \quad \leftrightarrow \quad R\text{-lattices} \quad \leftrightarrow \quad \text{compingent algebras.}$$

Classic Stone Duality

- Given a 0-dimensional compact Hausdorff space X, we consider its clopen sets S ordered by inclusion, i.e.

$$O \leq N \qquad \Leftrightarrow \qquad O \subseteq N.$$

- This poset S has a minimum $0 = \emptyset$ and a maximum $1 = X$ as well as meets, joins and complements given by

$$O \wedge N = O \cap N.$$
$$O \vee N = O \cup N.$$
$$O^c = X \setminus O.$$

- In fact, S is a Boolean algebra as it also satisfies

$$M \wedge (N \vee O) = (M \wedge N) \vee (M \wedge O). \qquad \text{(Distributivity)}$$

Theorem (Stone 1937)

Every Boolean algebra arises in this way.

Hofmann-Lawson Duality

- Given locally compact X, consider all open sets S again with

$$O \leq N \qquad \Leftrightarrow \qquad O \subseteq N.$$

- S is a complete lattice and in fact a frame, satisfying

$$O \wedge \bigvee \mathcal{N} = \bigvee_{N \in \mathcal{N}} O \wedge N. \qquad \text{(Infinite Distributivity)}$$

- Can recover compact containment $\Subset$ as way-below $\ll$:

$$O \Subset N \qquad \Leftrightarrow \qquad \exists \text{ compact } C \ (O \subseteq C \subseteq N).$$

$$O \ll N \qquad \Leftrightarrow \qquad O \in \bigcap \{ I \subseteq S : I \text{ is an ideal with } N \leq \bigvee I \}$$

- Moreover, S is continuous, i.e. for all O,

$$O = \bigvee_{N \ll O} N.$$

Theorem (Hoffman-Lawson 1978)

Every continuous frame arises in this way.

Pluses and Minuses

- (+) Continuous frames describe general locally compact spaces that are not 0-dimensional or even Hausdorff.
- (−) Spaces often have uncountably many open sets, even when they have countable bases, e.g. the Cantor space $\{0,1\}^{\mathbb{N}}$.
- (−) Frames are 2^{nd} order but Boolean algebras are 1^{st} order, i.e.
  - frames rely on joins of arbitrary infinite subsets while
  - Boolean algebras are defined by a finite list of 1^{st} order sentences involving finitary operations and relations (i.e. $\leq$).
- So classical model theory only applies Boolean algebras, e.g.
  1. Łoś's theorem: ultraproducts of Boolean algebras are Boolean.
  2. Löwenheim-Skolem theorem: arbitrarily large Boolean algebras have countable elementary submodels.
  3. Fraïsse theory: homogeneous Fraïsse limits exist for suitable subclasses of Boolean algebras.
- Goal: find a 1^{st} order extension of classic Stone duality to spaces that are not 0-dimensional.

Bases

- Given compact Hausdorff X, take a basis S of open sets
- Problem: $(S, \subseteq)$ does not always determine X.
- E.g. Consider the Boolean algebra B of regular open subsets of the unit inverval $[0, 1]$ with its usual topology.
- This has a countable basis, which generates a countable subalgebra S of B.
- X has no isolated points so S has no atoms.
- All countable atomless Boolean algebras are isomorphic.
- Thus $S \approx$ clopen subsets of the Cantor space $\{0, 1\}^{\mathbb{N}}$.
- i.e. arbitrary bases fail to distinguish $[0, 1]$ and $\{0, 1\}^{\mathbb{N}}$.
- Solution: either
  1. restrict to certain bases with enough (1^{st} order) conditions so compact Hausdorff spaces are distinguished, or
  2. add more structure, e.g. add compact containment as an extra primitive relation (i.e. $\ll$ is not defined by $\leq$)
- Shirota/De Vries does a bit of both. We go for option 1.

$\cup$-Bases

- Given compact Hausdorff X, consider a $\cup$-basis S of open sets, i.e. require S to be closed under finite unions

$$O, N \in S \qquad \Rightarrow \qquad O \cup N \in S \quad (\text{and } \emptyset = \bigcup \emptyset \in S).$$

- If every $O \in S$ is clopen then S must be the entire clopen lattice as in the classical Stone duality.
  (Proof: any clopen $C \subseteq X$ has a cover in S, as S is a basis, which can be made finite, as C is compact,
  so the union C is in S, as S is $\cup$-closed.)
- Problem: compact containment $\Subset$ is not way-below $\ll$.
- E.g. let $S =$ clopen subsets of the Cantor space $X = \{0,1\}^{\mathbb{N}}$.
- So $\Subset$ is just $\subseteq$. In particular, $X \Subset X$.
- Take any $x \in X$ and consider the ideal

$$I = \{O \in S : x \notin O\}.$$

- Then $\bigcup I = X \setminus \{x\}$. As x is not isolated $\bigvee I = X$ in S.
- But $X \notin I$ so $X \not\ll X$.

Rather Below

- Assumption: S is a $\cup$-basis of compact Hausdorff X.
- In particular, S is bounded: $\min(S) = \emptyset$ and $\max(S) = X$.
- Can then recover compact containment $\Subset$ as rather below:

$$O \Subset N \qquad \Leftrightarrow \qquad \exists M \in S \; (O \cap M = \emptyset \text{ and } N \cup M = X).$$

- Proof ($\Rightarrow$): As X is Hausdorff,

$$O \Subset N \qquad \Leftrightarrow \qquad \overline{O} \subseteq N.$$

  $\therefore$ for each $x \in X \setminus N$ we have $x \in O_x \in S$ with $O_x \cap O = \emptyset$. As $X \setminus N$ is compact, we have a finite subcover, whose union M is in S, as S is a $\cup$-basis. Note $O \cap M = \emptyset$ and $X = N \cup M$.
- Proof ($\Leftarrow$): If $O \cap M = \emptyset$ then $\overline{O} \cap M = \emptyset$.
  Thus $X = N \cup M$ implies $\overline{O} \subseteq N$.
- Can show that points of X correspond to maximal $\Subset$-filters.
- Thus X can be recovered from $(S, \subseteq)$.
- In particular, $\cup$-bases distinguish compact Hausdorff spaces.

Rather Distributive

- Goal: Characterize $\cup$-bases S of compact Hausdorff X (considering S as a poset where $\leq$ is $\subseteq$ and $\prec$ is $\Subset$)

Proposition

Any $\cup$-basis S satisfies $\prec$-distributivity:

$$a \ \leq \ b \vee c \quad \Leftrightarrow \quad \forall a' \prec a \ \exists b' \prec b \ \exists c' \prec c \ (a' \ \prec \ b' \vee c' \ \prec \ a).$$

- Note $\leq$-distributivity reduces to $a' = a$ case, i.e.

$$a \ \leq \ b \vee c \quad \Leftrightarrow \quad \exists b' \leq b \ \exists c' \leq c \ (a \ = \ b' \vee c'),$$

  which is the usual notion of distributivity for $\vee$-semilattices.

Theorem (B.-Starling 2017)

Every (bounded) $\prec$-distributive $\vee$-semilattice arises in this way.

- Classic Stone duality recovered when $\prec$ is reflexive/$\prec = \leq$.

Generalizations

- Given locally compact Hausdorff X, consider a $\cup$-basis S of relatively compact open sets. Then S has no maximum but

$$\forall O \in S \;\; \exists N \in S \;\; (O \Subset N). \qquad (\Subset\text{-round})$$

$$O \Subset N \quad \Leftrightarrow \quad \forall P \supseteq N \; \exists M \in S \; (O \cap M = \emptyset \text{ and } N \cup M \supseteq P).$$

Theorem (B.-Starling 2017)

Every $\prec$-round $\prec$-distributive $\vee$-semilattice (with 0) arises this way.

- For locally compact locally Hausdorff X, take a basis $S \ni \emptyset$:

$$\exists \text{ open Hausdorff } M \Supset O, N \in S \quad \Leftrightarrow \quad O \cup N \in S. \quad (\cup)$$

- Note $\Leftarrow$ means every $O \in S$ is compactly contained in some open Hausdorff set (in particular O itself is Hausdorff).
- Such a $\cup$-basis S is only a conditional $\vee$-semilattice, i.e.

$$a, b \leq c \qquad \Rightarrow \qquad a \vee b \text{ exists.}$$

- Problem: being a conditional $\vee$-semilattice is not sufficient to characterize such $\cup$-bases.
- Solution: find a way to express $(\cup)$ order theoretically.

The Hausdorff Relation

- Given a poset S, define the Hausdorff relation $\smile$ by

$$a \smile b \qquad \Leftrightarrow \qquad \forall a' \prec a\, \forall b' \prec b\, \exists c \prec a, b\, \forall c' \leq a', b'\, (c' \prec c).$$

- If S is a $\prec$-distributive $\wedge$-semilattice then

$$a \smile b \qquad \Leftrightarrow \qquad \forall a' \prec a \;\; \forall b' \prec b \;\; (a' \wedge b' \;\; \prec \;\; a \wedge b).$$

Proposition

If S is a $\cup$-basis of some locally compact locally Hausdorff X then

$$O \cup N \text{ is Hausdorff} \qquad \Leftrightarrow \qquad O \smile N \text{ holds in } S.$$

Theorem (B.-Starling 2017)

Every $\prec$-round $\prec$-distributive poset (with minimum 0) satisfying

$$a \prec a' \smile b' \succ b \qquad \Rightarrow \qquad a \vee b \text{ exists}$$

comes from a $\cup$-basis of some locally compact locally Hausdorff X.

Non-Commutative Extension

- Given locally compact locally Hausdorff etalé groupoid G, consider a basis S of open Hausdorff bisections such that

$$O \in S \quad \Rightarrow \quad O^{-1} \in S.$$
$$O, N \in S \quad \Rightarrow \quad O \cdot N \in S.$$

$\exists$ open Hausdorff bisection $M \supseteq O, N \in S \quad \Leftrightarrow \quad O \cup N \in S.$

- S is a $\prec$-round $\prec$-distributive inverse semigroup with 0 and

$$a \prec a' \underset{\smile}{\sim} b' \succ b \qquad \Rightarrow \qquad a \vee b \text{ exists}$$
$$a \prec b \qquad \Rightarrow \qquad a^{-1}a \prec b^{-1}b.$$

Theorem (B.-Starling 2017)

Every such inverse semigroup arises in this way.

- Final note: these dualities are also functorial

etalé groupoids	$\leftrightarrow$	inverse semigroups.
continuous functors	$\leftrightarrow$	relational morphisms.