The Balian-Low Theorem and Vector Bundles

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Work in progress with Franz Luef

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- M. Rieffel/F. Luef: Projective modules over noncommutative tori.

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Definition

$\{\eta_{k,l}\}_{k,l \in \mathbb{Z}}$ is called a *Gabor frame* if there exist constants $C, D \geq 0$ such that

$$C \|\xi\|^2 \leq \sum_{k,l \in \mathbb{Z}} |\langle \xi, \eta_{k,l} \rangle|^2 \leq D \|\xi\|^2$$

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- For every $\xi \in L^2(\mathbb{R})$, we have that

$$\xi = \sum_{k,l \in \mathbb{Z}} \langle S^{-1}\xi, \eta_{k,l} \rangle \eta_{k,l}.$$

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- Let $\xi(x) = e^{-\pi x^2}$ and $\eta = \chi_{[0,1]}$. Then $\{\xi_{k,l}\}_{k,l}$ is a Gabor frame for all $0 < \alpha\beta < 1$ but not $\alpha\beta = 1$. $\{\eta_{k,l}\}_{k,l}$ is a Gabor frame for $\alpha\beta = 1$.

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Theorem (R. Balian, F. Low)

Let $\eta \in L^2(\mathbb{R})$. Suppose that

1. $\alpha\beta = 1$.
2. η is "nice": $\eta \in S_0(\mathbb{R})$.

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- Moral: When working with Gabor frames at the critical density $\alpha\beta = 1$, we cannot expect well-behaved Gabor atoms η.

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- For $x \in G$ and $\omega \in \widehat{G}$, define operators on $L^2(G)$ by

$$T_x\xi(t) = \xi(x^{-1}t) \qquad\qquad M_\omega\xi(t) = \omega(t)\xi(t).$$

Set $\pi(x,\omega) = M_\omega T_x$.

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- Given a discrete set $\Delta \subseteq G \times \widehat{G}$ and $\eta \in L^2(G)$, we call

$$\mathcal{G}(\eta, \Delta) = \{\pi(z)\eta : z \in \Delta\}$$

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- If there exist $C, D \geq 0$ such that

$$C\|\xi\|^2 \leq \sum_{z\in\Delta} |\langle \xi, \pi(z)\eta\rangle|^2 \leq D\|\xi\|^2$$

  for all $\xi \in L^2(G)$, then $\mathcal{G}(\eta, \Delta)$ is called a *Gabor frame*.

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Theorem?

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Example (Failure of general BL)

Let D be a discrete group and set $G = D \times \widehat{D}$, $H = D \times \{e\}$. Set $\eta = \chi_{\{e\} \times \widehat{D}}$. Then $\eta \in S_0(G)$, and $\mathcal{G}(\eta, H \times H^\perp)$ is an orthonormal basis for $L^2(G)$.

The Zak Transform

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- For $\xi \in L^2(G)$, the function $Z_H\xi : G \times \widehat{G} \to \mathbb{C}$ given by
$$Z_H\xi(x,\omega) = \sum_{h \in H} \xi(h^{-1}x)\omega(h).$$
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- The Zak transform $F = Z_H\xi$ satisfies the following *quasiperiodic* property:

$$F(xh,\omega\tau) = \overline{\tau(h)}F\xi(x,\omega) \qquad ; \quad x \in G, \omega \in \widehat{G}, h \in H, \tau \in H^{\perp}. \quad (1)$$

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- Gröchenig 1998: $\mathcal{G}(\eta, H \times H^{\perp})$ is a Gabor frame if and only if there exist $0 < A, B < \infty$ such that

$$A \leq |Z_H\eta(x,\omega)| \leq B$$

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- If $\xi \in S_0(G)$, then $Z_H\xi$ is continuous and bounded. Hence, $\mathcal{G}(\eta, H \times H^\perp)$ is NOT a Gabor frame if and only if $Z_H\xi$ has a zero.

The vector bundle

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- Denote by $E = E_{G,H}$ the quotient of $G \times \widehat{G} \times \mathbb{C}$ by the equivalence relation generated by

$$(x, \omega, \lambda) \sim (xh, \omega\tau, \omega(h)\lambda) \quad ; \quad x \in G, \omega \in \widehat{G}, h \in H, \tau \in H^{\perp}.$$

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Theorem

Let G be a locally compact Abelian group, and let H be a uniform lattice in G. Then the vector bundle E is nontrivial if and only if for every $\eta \in L^2(G)$ the following implication holds: Whenever $Z_H\eta$ is continuous, then $\mathcal{G}(\eta, H \times H^{\perp})$ is not a Gabor frame.

Twisted group C^*-algebras

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- The map $\pi(x,\omega) = M_\omega T_x$ satisfies

$$\pi(x,\omega)\pi(y,\tau) = \overline{\tau(x)}\pi(xy,\omega\tau).$$

where $c((x,\omega),(y,\tau)) = \overline{\tau(x)}$ is a 2-cocycle on $G \times \widehat{G}$.

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- Given a discrete subgroup $\Delta \subseteq G \times \widehat{G}$, let $A_0 = L^1(\Delta, c)$ be the c-twisted convolution algebra with operations

$$(f * g)(z) = \sum_{w \in \Delta} f(w)g(w^{-1}z)c(w,w^{-1}z) \quad f^*(z) = \overline{c(z,z^{-1})f(z^{-1})}$$

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- Let $A = C^*(\Delta, c)$ be the corresponding c-twisted group C^*-algebra.

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- Let $A = C^*(\Delta, c)$ be the corresponding c-twisted group C^*-algebra.
- We obtain the integrated representation $\Pi : C^*(\Delta, c) \to \mathscr{B}(L^2(G))$ given by

$$\Pi(f) = \sum_{z \in \Delta} f(z)\pi(z).$$

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$$f \cdot \xi = \Pi(f)\xi$$
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give $S_0(G)$ the structure of a pre-inner product A_0-module $\mathcal{E}_0$. Its completion $\mathcal{E}$ is a finitely generated projective left Hilbert A-module.

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  give $S_0(G)$ the structure of a pre-inner product A_0-module $\mathcal{E}_0$. Its completion $\mathcal{E}$ is a finitely generated projective left Hilbert A-module.
- Trace $\mathrm{tr} : A \to \mathbb{C}$ given on A_0 by $\mathrm{tr}(f) = f(e)$.

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A *module frame* in $\mathcal{E}$ is a sequence $(\eta_j)_{j\in J}$ in $\mathcal{E}$ such that there exist constants $C, D \geq 0$ with

$$C {}_\bullet\langle \xi, \xi \rangle \leq \sum_{j\in J} {}_\bullet\langle \xi, \eta_j \rangle^* {}_\bullet\langle \xi, \eta_j \rangle \leq D {}_\bullet\langle \xi, \xi \rangle$$

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for all $\xi \in \mathcal{E}$.

- Restrict to a one-element module frame $\{\eta\}$. Taking traces, we obtain

$$C\|\xi\|^2 \leq \sum_{z\in\Delta} |\langle \xi, \pi(z)\eta\rangle|^2 \leq D\|\xi\|^2.$$

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- Thus, $\mathcal{E}$ is a finitely generated projective module over the algebra of continuous functions on $G/H \times \widehat{G}/H^{\perp}$.
- By the Serre-Swan theorem, there exists a Hermitian vector bundle $E = E_{G,H} \to G/H \times \widehat{G}/H^{\perp}$ such that $\mathcal{E} \cong \Gamma(E)$ as left Hilbert C^*-modules.

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- Equip $\Gamma(E)$ with the following $C(G/H \times \widehat{G}/H^{\perp})$-module structure and inner product: [Rieffel 1983: The case $G = \mathbb{R}$]

$$(f \cdot F)(x, \omega) = f([x], [\omega]) F(x, \omega)$$

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Module-theoretic proof of BL variant

Suppose $\eta \in S_0(G)$, and let H be a uniform lattice in G. Then if $E_{G,H}$ is nontrivial, then $\{\eta\}$ cannot be a module frame for $\mathcal{E}_{G,H}$.

Generalizations

- Investigate the situation where H is not necessarily a uniform lattice in G.
- Investigate more subgroups $\Delta \subset G \times \widehat{G}$ for which $C^*(\Delta, c)$ is commutative.

Thank you for your attention!