WWU MÜNSTER

Gelfand pairs, spherical functions and exotic group C^*-algebras

Tim de Laat

(Joint work with Dennis Heinig and Timo Siebenand)

March 29, 2021

Universal and reduced group C^*-algebra

G – locally compact group

- $\|f\|_r = \|\lambda(f)\|_{\mathcal{B}(L^2(G))}$, where $\lambda\colon G \to \mathcal{U}(L^2(G))$, $(\lambda(s)\xi)(t) = \xi(s^{-1}t)$.
- $C^*_r(G) := \overline{C_c(G)}^{\|\cdot\|_r}$ – **reduced group C^*-algebra**.

- $\|f\|_u = \sup\{\|\pi(f)\| \mid \pi \text{ un. rep. of } G\}$.
- $C^*(G) := \overline{C_c(G)}^{\|\cdot\|_u}$ – **universal group C^*-algebra**.

Theorem

$C^*(G) = C^*_r(G)$ if and only if G is **amenable**.

Exotic group C^*-algebras

Definition

A **Group C^*-algebra** of G is a completion $C^*_\mu(G)$ of $C_c(G)$ w.r.t. a C^*-norm $\|.\|_\mu$ satisfying $\|f\|_u \geq \|f\|_\mu \geq \|f\|_r$ for all $f \in C_c(G)$.

Identity map on $C_c(G)$ induces canonical surjective *-homomorphisms

$$C^*(G) \twoheadrightarrow C^*_\mu(G) \twoheadrightarrow C^*_r(G).$$

The algebra $C^*_\mu(G)$ is **exotic** if both quotient maps are not injective.

Question: Does every non-amenable G have exotic group C^*-algebras?

This question is still open!

Why are they interesting?

- Reflect structural properties of the underlying group.
- **Crossed product functors** can be constructed from exotic group C^*-algebras, in particular from exotic group C^*-algebras whose dual space is a G-invariant weak*-closed ideal of $B(G)$.
  Relates to **Baum–Connes conjecture** with coefficients.

What is known for discrete groups?

First systematic approach: **Ideal completions** [Brown–Guentner (2013)].

Γ countable group, D alg. two-sided ideal of $\ell^\infty(\Gamma)$. The ideal completion $C^*_D(\Gamma)$ is the completion of the group ring with respect to the norm defined through all un. reps with many matrix coefficients in D. (Matrix coefficient: $\pi_{\xi,\zeta} : s \mapsto \langle \pi(s)\xi, \zeta \rangle$.)

- $C^*_{\ell^p}(\mathbb{F}_d) \twoheadrightarrow C^*_{\ell^q}(\mathbb{F}_d)$ is not injective if $2 \leq q < p \leq \infty$ [Okayasu (2014)]. Same for discrete groups Γ with $\mathbb{F}_d < \Gamma$ [Wiersma (2016)].
- Other constructions and examples, many results due to Wiersma.

What about Lie groups?

- $C^*_{L^p}(\mathrm{SL}(2,\mathbb{R})) \twoheadrightarrow C^*_{L^q}(\mathrm{SL}(2,\mathbb{R}))$ is not injective if $2 \leq q < p \leq \infty$ [Wiersma (2015)]. Analysis of the unitary dual.
- Same for $\mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$ [Samei–Wiersma (2018)]. Relies on the Kunze – Stein property and a strengthened version of the Haagerup property.
- For $\mathrm{SL}(2,\mathbb{R})$: Only ones coming from G-invariant ideals of $B(G)$.

They are not so exotic after all.

Questions: Can we understand these results more directly in terms of representations? What about Lie groups with property (T)? What about other classes of locally compact groups?

L^p-integrability of matrix coefficients

Construct exotic group algebras of locally compact groups through L^p-integrability properties of matrix coefficients.

Definition

Let $p \in [2, \infty]$. A un. rep. $\pi \colon G \to \mathcal{U}(\mathcal{H})$ is an L^p-representation if there is a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\pi_{\xi,\zeta} \in L^p(G)$ for all $\xi, \zeta \in \mathcal{H}_0$. π is an L^{p+}-representation if π is $L^{p+\varepsilon}$ for all $\varepsilon > 0$.

($f \in C_b(G) \cap L^p(G)$ implies: f is contained in $L^q(G)$ for all $q \geq p$.)

L^{p+}-representations behave well w.r.t. weak containment.

The algebras $C^*_{L^{p+}(G)}$

G locally compact group, $p \in [2, \infty]$.

$C^*_{L^p}(G)$ and $C^*_{L^{p+}}(G)$ are the completions of $C_c(G)$ with respect to

$$\|\cdot\|_{L^p} : C_c(G) \to [0, \infty),\ f \mapsto \sup\{\|\pi(f)\| \mid \pi \text{ is an } L^p\text{-rep.}\} \text{ and}$$
$$\|\cdot\|_{L^{p+}} : C_c(G) \to [0, \infty),\ f \mapsto \sup\{\|\pi(f)\| \mid \pi \text{ is an } L^{p+}\text{-rep.}\}.$$

The Kunze–Stein property

G is a **Kunze–Stein group** if $m\colon C_c(G) \times C_c(G) \to C_c(G),\ (f,g) \mapsto f * g$ extends to a bounded bil. map $L^q(G) \times L^2(G) \to L^2(G)$ for all $q \in [1,2)$.

Examples:

- $\mathrm{SL}(2,\mathbb{R})$ [Kunze–Stein (1960)]
- Connected semisimple Lie groups with finite center [Cowling (1978)] and non-Archimedean analogues [Veca (2002)]
- Groups of automorphisms of trees [Nebbia (1988)]

If G is non-compact and amenable and m extends to a bounded bil. map $L^q \times L^2 \to L^2(G)$, then $q = 1$.

L^{p+}-representations of Kunze–Stein groups

For $p \in [2, \infty]$, set

$$\widehat{G}_{L^{p+}} := \{[\pi] \in \widehat{G} \mid \pi \text{ is an } L^{p+}\text{-representation}\}$$

Theorem [dL – Siebenand (2019)]

Let G be a Kunze-Stein group. Then $\widehat{G}_{L^{p+}}$ is Fell-closed in $\widehat{G}$.

($\overline{S}$ consists of all $[\pi]$ in $\widehat{G}$ which are weakly contained in S.)

Was known for $\mathrm{SO}_0(n, 1)$ and $\mathrm{SU}(n, 1)$ from work of Shalom (2000).

Gelfand pairs

Let G be a locally compact group with a compact subgroup K.
$\varphi \colon G \to \mathbb{C}$ is K-bi-invariant if $\varphi(k_1 s k_2) = \varphi(s)$ for all $s \in G$ and $k_1, k_2 \in K$.

Definition

The pair (G, K) is a **Gelfand pair** if the *-subalgebra $C_c(K \backslash G / K)$ of $C_c(G)$ (consisting of all K-bi-invariant elements) is commutative.

Examples of Gelfand pairs:

- Simple (more generally reductive) Lie groups G with maximal compact subgroup K.
- Certain classes of non-compact, closed subgroups G of $\mathrm{Aut}(T)$, where T is a nice tree, with K a vertex stabiliser subgroup.

Spherical functions and class one representations

Definition

For a Gelfand pair (G,K), a K-bi-invariant $\varphi \in C(G)$ with $\varphi(e) = 1$ such that the map $C_c(K\backslash G/K) \to \mathbb{C}$, $f \mapsto \int f(s)\varphi(s^{-1})\mathrm{d}\mu_G(s)$ forms an algebra homomorphism is called a **spherical function** for (G,K).

$\pi \in \hat{G}$ is **class one** for (G,K) if the vector space $\mathcal{H}^K$ of K-invariant vectors is one-dimensional. $\rightsquigarrow (\hat{G}_K)_1$.

(G,K) is a Gelfand pair $\rightsquigarrow$ spherical functions are diagonal matrix coefficients $\pi_{\xi,\xi}$, with $\xi \in \mathcal{H}^K \backslash \{0\}$.

Simple Lie groups

Analyze the asymptotics and L^p-integrability of spherical functions.

G connected simple Lie group $\rightsquigarrow G = K\overline{A^+}K$,
K maximal compact subgroup, A abelian

Real rank of $G = \dim(\mathrm{Lie}(A))$

Simple Lie groups with real rank one

G – connected simple Lie group with real rank one.
Then G is locally isomorphic to one of the following Lie groups:

$$\mathrm{SO}(n,1) = \{g \in \mathrm{SL}(n+1,\mathbb{R}) \mid g^* I_{n,1} g = I_{n,1}\},$$
$$\mathrm{SU}(n,1) = \{g \in \mathrm{SL}(n+1,\mathbb{C}) \mid g^* I_{n,1} g = I_{n,1}\},$$
$$\mathrm{Sp}(n,1) = \{g \in \mathrm{GL}(n+1,\mathbb{H}) \mid g^* I_{n,1} g = I_{n,1}\},$$
$$\mathrm{F}_{4(-20)}.$$

First three: Isometry groups of the classical rank one symmetric spaces of the non-compact type. Class one representation theory is well understood.

Locally compact group G:

$$\Phi(G) := \inf\{p \in [1,\infty] \mid \forall \pi \in \widehat{G}\backslash\{\tau_0\}, \pi \text{ is an } L^{p+}\text{-representation}\},$$

where τ_0 is the trivial representation.

For the classical real rank one Lie groups:

$$\Phi(G) = \begin{cases} \infty & \text{if } G = \mathrm{SO}_0(n,1), \\ \infty & \text{if } G = \mathrm{SU}(n,1), \\ 2n+1 & \text{if } G = \mathrm{Sp}(n,1). \end{cases}$$

First two cases: Harish-Chandra
$\mathrm{Sp}(n,1)$: Li (1995).

Theorem [dL – Siebenand (2019)]

Let G be a classical simple Lie group with real rank one and finite center. Then for $2 \leq q < p \leq \Phi(G)$, the canonical quotient map

$$C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$$

has non-trivial kernel. Furthermore, for every $p, q \in [\Phi(G), \infty)$, we have

$$C^*_{L^{p+}}(G) = C^*_{L^{q+}}(G).$$

Partial results for $F_{4(-20)}$. For finite coverings, the result is the same.

Samei and Wiersma (2018) already covered $SO_0(n,1)$ and $SU(n,1)$.

About the proof

- By the earlier theorem, the first claim reduces to finding L^{p+}-representations which are not L^{q+} for $2 \leq q < p \leq \Phi(G)$.
- Consider the strip of class one complementary series representations.
- Asymptotics (L^p-integrability) of spherical functions follows from Harish-Chandra's rich work. One can realize all necessary L^p-integability in this strip.
- The second claim follows from a result of Cowling (1979). (Quantitative version of property (T).)

Locally compact groups acting on trees

Similar methods give:

Theorem [Heinig – dL – Siebenand (2020)]

Let T be a semi-homogeneous tree of degree (d_0, d_1) with $d_0, d_1 \geq 2$ and $d_0 + d_1 \geq 5$, and let G be a non-compact, closed subgroup of the automorphism group $\mathrm{Aut}(T)$. Suppose that G acts transitively on the boundary ∂T. For $2 \leq q < p \leq \infty$, the canonical quotient map

$$C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$$

is not injective.

A uniqueness result

Theorem [Heinig – dL – Siebenand (2020)]

Let T be a **homogeneous** tree of degree $d \geq 3$, and let G be a non-compact, closed subgroup of the automorphism group $\mathrm{Aut}(T)$. Suppose that G acts **transitively on the vertices of T** and on the boundary ∂T and that G satisfies **Tits' independence property**. If $C^*_\mu(G)$ is a group C^*-algebra of G such that its dual space $C^*_\mu(G)^*$ is a G-invariant ideal of $B(G)$, then there exists a unique $p \in [2, \infty]$ such that $B_{L^{p+}}(G) = C^*_\mu(G)^*$, where $B_{L^{p+}}(G) := C^*_{L^{p+}}(G)^*$.

About the proof I

- Suppose $C^*_\mu(G)^*$ is a G-invariant weak*-closed ideal in $B(G)$.
- Let $o \in V(T)$ and $K = G_o$. We know $\tilde{G}\backslash\tilde{G}_r \subset (\tilde{G}_K)_1$.
- Let $C^*_\mu(K\backslash G/K)$ be the completion of $C_c(K\backslash G/K)$ in $C^*_\mu(G)$. It is a commutative sub-C*-algebra of $C^*_\mu(G)$ whenever (G, K) is a Gelfand pair.
- By K-amenability, the canonical quotient map $s \colon C^*_\mu(G) \to C^*_r(G)$ induces an isomorphism $s_* \colon K_i(C^*_\mu(G)) \to K_i(C^*_r(G))$ for $i \in \{0, 1\}$. This implies that the canonical quotient map $s| \colon C^*_\mu(K\backslash G/K) \to C^*_r(K\backslash G/K)$ induces an isomorphism $(s|)_* \colon K_i(C^*_\mu(K\backslash G/K)) \to K_i(C^*_r(K\backslash G/K))$ for $i \in \{0, 1\}$.

About the proof II

- The Gelfand transformation gives a *-isomorphism from $C^*_\mu(K\backslash G/K)$ to $C(\sigma_{C^*_\mu(K\backslash G/K)}(\mu_1))$ (where μ_1 is a generating self-adjoint element of $C_c(K\backslash G/K)$). Let $r \in \sigma_{C^*_\mu(K\backslash G/K)}(\mu_1)$ be the largest number in $\sigma_{C^*_\mu(K\backslash G/K)}(\mu_1)$. Then $\sigma_{C^*_\mu(K\backslash G/K)}(\mu_1) \subset [-r, r]$ and $-r \in \sigma_{C^*_\mu(K\backslash G/K)}(\mu_1)$.
- There is an element $p \in [2, \infty]$ with $\sigma_{C^*_{L^{p+}}(K\backslash G/K)}(\mu_1) = [-r, r]$, so it suffices to show that $\sigma_{C^*_\mu(K\backslash G/K)}(\mu_1) = [-r, r]$.
  (This will be omitted.)

Concluding remark

- Heinig – dL – Siebenand (2020): Let T be a semi-homogeneous tree, G a non-compact, closed subgroup of $\mathrm{Aut}(T)$. Suppose that G acts transitively on the boundary ∂T and that G has Tits' independence property. Then every group C^*-algebra $C^*_\mu(G)$ that is distinguishable (as a group C^*-algebra) from $C^*(G)$ and whose dual space is a G-invariant ideal of $B(G)$ is **abstractly *-isomorphic** to the reduced group C^*-algebra of G.