An algebraic characterization of type I ample groupoids

Gabriel Favre joint with S. Raum

Stockholm University

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We want to understand algebraically the unitary representations of groupoid $\mathcal{G}$ as whole, namely the type I property.

- van Wyk (2017) gave a topological criterion of the type I property for ample groupoids in terms of the isotropy groups and in terms of the separation of the orbit space.
- Lawson and Lenz (2010) gave a Stone type duality between ample groupoids $\mathcal{G}$ and their boolean inverse semigroups of compact open bisections $\Gamma(\mathcal{G})$. This duality doesn't consider all natural groupoid morphisms.
- We used both these results to give a characterization involving quotients of $\Gamma(\mathcal{G})$ of the type I property for $\mathcal{G}$.

Left regular representation

G will always denote a countable discrete group.

- For a Hilbert space H, write $B(H)$ for the space of all bounded linear operators on H.
- A *(unitary) representation* (π, H) of a discrete group G is the data of a Hilbert space H and a group homomorphism $\pi : G \to U(H)$.
- For $g \in G$, $\lambda_g \in B(L^2(G))$ defined by the formula

$$(\lambda_g(f))(h) = f(g^{-1}h),\ f \in L^2(G),\ h \in G,$$

is unitary. The left regular representation λ of G extends to the complex group ring

$$\lambda : \mathbb{C}[G] \to B(L^2(G)).$$

Type I von Neumann algebras

The group von Neumann algebra $L(G)$ of G is the strong closure of the image of λ.

$$\mathrm{L}(G) = \overline{\lambda(\mathbb{C}[G])}^{\mathrm{SOT}} \subseteq B(L^2(G)).$$

Definition (due to a result of Smith)

A discrete group G is *type I* if its group von Neumann algebra can be decomposed as a direct sum

$$\mathrm{L}(G) \cong (L^\infty(X_\infty) \otimes B(\ell^2(\mathbb{N}))) \oplus \bigoplus_{n=0}^{\infty} L^\infty(X_n) \otimes M_n(\mathbb{C}),$$

for some measure spaces X_n.

Type I groups, examples cont.

- Finite groups.
- Abelian groups.
- Let G be a countable discrete group which has an abelian subgroup $A \leq G$ of finite index n. There is an inclusion

$$L(G) \subseteq L(A) \otimes M_n(\mathbb{C}).$$

Theorem (Thoma 1968)

A discrete group G is type I if and only if it is virtually abelian.

Type I groupoids

Theorem (van Wyk 2017)

A second countable, ample groupoid $\mathcal{G}$ is type I if and only if its orbits space is (T_0) and its isotropy groups are type I.

Thanks to the result of Thoma, and the Ramsey-Effros-Mackey dichotomy, one can reformulate the theorem of van Wyk slightly as follows.

Theorem (van Wyk 2017)

A second countable, ample groupoid $\mathcal{G}$ is type I if and only if its orbits are *locally closed* and its isotropy groups are *virtually abelian*.

Type I groupoids, example and non-examples

- Consider the one point compactification $\widehat{\mathbb{N}} = \{\infty\} \cup \mathbb{N}$ of the natural numbers, and say $x \sim y$ if $x, y < \infty$. The groupoid $\mathcal{G}$ associated to the equivalence relation $\sim$ has 2 orbits, $\mathbb{N}$ and $\{\infty\}$, hence the orbit space is a 2 points space. The singleton $\{\infty\}$ is open in the orbit space, so that $\mathcal{G}^{(0)}/\mathcal{G}$ is (T_0).

- Consider the space $X := \{0,1\}^{\mathbb{N}}$ of infinite sequences of 0 and 1 with the product topology. Say that two sequence are equivalent if they eventually agree. The orbit of every point is dense.
- The transformation groupoid associated with the action of $SL_2(\mathbb{Z})$ on the torus $\mathbb{T}^2$ is not type I, since the orbit of irrational points are dense.

Ample groupoids

Groupoids will be denoted by $\mathcal{G}$. We will denote the set of units of a groupoid $\mathcal{G}$ by $\mathcal{G}^{(0)} \subseteq \mathcal{G}$, the source and range maps by

$$d, r : \mathcal{G} \to \mathcal{G}^{(0)}.$$

d and r are continuous.

An *open bisection* is an open set $U \subseteq \mathcal{G}$ such that $d|_U$ and $r|_U$ are both homeomorphisms onto their image.

Groupoids will always be *ample, locally compact Hausdorff* and *second countable*.

Orbit space, isotropy groups

Given a unit $x \in \mathcal{G}^{(0)}$, $\mathcal{G}_x := d^{-1}(x) \cap r^{-1}(x)$, is the *isotropy group* of x. The set

$$\mathcal{G}x := \{y \in \mathcal{G}^{(0)} \mid \exists \gamma \in \mathcal{G}, r(\gamma) = y, d(\gamma) = x\},$$

is the *orbit* of x.

Proposition (Ramsey 1990)

Let $\mathcal{G}$ be a second countable Hausdorff ample groupoid. The following are equivalent.

- the orbit space of $\mathcal{G}$ is (T_0);
- the orbits of $\mathcal{G}$ are locally closed;
- there are no self-accumulating orbits.

Compact open bisections of an ample groupoid

Let $\Gamma(\mathcal{G})$ the set of all the compact open bisections of a groupoid $\mathcal{G}$. Given two bisections U, V, one can define their product

$$UV := \{uv \mid d(u) = r(v)\},$$

and a generalized inverse $U^* := \{u^{-1} \mid u \in U\}$.

$\Gamma(\mathcal{G})$ does not have an identity if $\mathcal{G}^{(0)}$ is not compact. The generalized inverse U^* of U is however the unique bisection satisfying the relations

$$UU^*U = U \text{ and } U^*UU^* = U^*.$$

The structure of $\Gamma(\mathcal{G})$

The set of idempotents E of $\Gamma(\mathcal{G})$ is commutative and has an can be given a poset structure by the formula $U \leq V$ if $UV = U$.

E a *lattice* structure. It is distributive and has a least element. This gives E the structure of a *generalized boolean algebra*.

Definition

A *generalized boolean algebra* is a distributive lattice with a 0 element.

Boolean inverse semigroups

In general, the set of idempotents of a semigroup is only a meet-semilattice.

Definition

A *boolean inverse semigroup* is an inverse semigroup S such that its set of idempotents $E(S) = \{ss^* \mid s \in S\}$ is a generalized boolean algebra, where $e \leq f$ if $ef = e$ for $e, f \in E(S)$.

Compact open bisections $\Gamma(\mathcal{G})$ of a groupoid $\mathcal{G}$ are the prime examples of boolean inverse semigroups.

Noncommutative Stone duality

Given a generalized boolean algebra B, the *spectrum* $\widehat{B}$ of B is a locally compact totally disconnected Hausdorff space. The noncommutative Stone duality generalizes the duality between generalized boolean algebras and totally disconnected spaces.

Given a boolean inverse semigroup B, the set of ultrafilters for the natural order on B forms a groupoid $\mathcal{G}(B)$.

Theorem (Lawson 2010+2012 Lawson-Lenz 2013)

The constructions $B \to \mathcal{G}(B)$, $\mathcal{G} \to \Gamma(\mathcal{G})$ are inverse to each other.

Corners and subgroupoids

Idempotents of $\Gamma(\mathcal{G})$ correspond to compact open subsets of $\mathcal{G}^{(0)}$. Given an idempotent $p \in \Gamma(\mathcal{G})$, the *corner* $p\Gamma(\mathcal{G})p$ is naturally isomorphic with $\Gamma(\mathcal{G}|_U)$, where U is the compact open of $\mathcal{G}^{(0)}$ corresponding to p.

There is a one-to-one correspondence between open subgroupoids of $\mathcal{G}$ and Boolean inverse subsemigroups $B \subseteq \Gamma(\mathcal{G})$ with $E(B) = E(\Gamma(\mathcal{G}))$. On one hand by the inclusion $\Gamma(\mathcal{H}) \subseteq \Gamma(\mathcal{G})$ for open subgroupoids $\mathcal{H} \subseteq \mathcal{G}$ and on another hand assigning the groupoid $\bigcup B \subseteq \mathcal{G}$ to Boolean inverse semigroups $B \subseteq \Gamma(\mathcal{G})$ such that $E(B) = E(\Gamma(\mathcal{G}))$.

Restriction morphisms

The following proposition ensures the compatiblity of the duality with restriction maps.

Proposition (Lawson-Vdovina 2019)

Let $\mathcal{G}$ be an ample groupoid, $A \subseteq \mathcal{G}^{(0)}$ a closed $\mathcal{G}-$invariant set. Then the restriction map $res_A : CO(\mathcal{G}^{(0)}) \to CO(A)$ extends to a unique homomorphism $\Gamma(\mathcal{G}) \to \Gamma(\mathcal{G}|_A)$ with the universal property that for every morphism $\pi : \Gamma(\mathcal{G}) \to B$ such that

$$\begin{array}{ccc} CO(\mathcal{G}^{(0)}) & \xrightarrow{\pi|_{CO(\mathcal{G}^{(0)})}} & E(B) \\ \downarrow{\scriptstyle res_A} & \nearrow & \\ CO(A) & & \end{array} \quad \text{extends uniquely to} \quad \begin{array}{ccc} \Gamma(\mathcal{G}) & \xrightarrow{\pi} & B \\ \downarrow{\scriptstyle res_A} & \nearrow & \\ \Gamma(\mathcal{G}|_A) & & \end{array}$$

Minimal groupoids and 0-simplifying Boolean inverse semigroups

An ample groupoid $\mathcal{G}$ is *minimal* if its orbits $\mathcal{G}x$ are dense. Let's introduce the corresponding semigroup notion.

Let B be a boolean inverse semigroup. The kernels $\pi^{-1}(0) \subseteq B$ of morphisms of boolean inverse semigroups $\pi : B \to C$ are called *additive ideals*. We say that B is *0-simplifying* if its only additive ideals are 0 and B.

Proposition (Steinberg-Szakács 2020)

Let $\mathcal{G}$ be an ample Hausdorff groupoid, then $\mathcal{G}$ is minimal if and only if $\Gamma(\mathcal{G})$ is 0-simplifying.

Minimal groupoids with compact, infinite unit space which have a self-accumulating orbit.

A small dictionnary

The noncommutative Stone duality establishes the following dictionnary.

Ample groupoid $\mathcal{G}$	Associated semigroup $\Gamma(\mathcal{G})$
$\mathcal{G}^{(0)}$ compact	$\Gamma(\mathcal{G})$ unital
$\mathcal{G}$ effective	$\Gamma(\mathcal{G})$ fundamental
$\mathcal{G}$ principal	$\Gamma(\mathcal{G})$ fundamental
$\mathcal{G}$ minimal	$\Gamma(\mathcal{G})$ 0-simplifying
$\mathcal{G}$ is a union of abelian groups	$\Gamma(\mathcal{G})$ abelian
$\mathcal{G}$ type I	slide 20
$\mathcal{G}$ CCR	similar to type I
$\mathcal{G}$ amenable	???

Seperation of the orbit space.

From van Wyk's theorem, the first ingredient we need to understand for characterizing the type I property is the separation property of the orbit space. We obtained the following result.

Proposition (Raum - F.)

Let $\mathcal{G}$ be a second countable, ample Hausdorff groupoid. Then the following statements are equivalent.

- The orbit space of $\mathcal{G}$ is not (T_0).
- A corner of $\Gamma(\mathcal{G})$ has an infinite, monoidal and 0-simplifying quotient.
- There is an infinite, monoidal and 0-simplifying subquotient of $\Gamma(\mathcal{G})$.

Corners of $\Gamma(\mathcal{G})$ are of the form $p\Gamma(\mathcal{G})p$, where p is an idempotent of $\Gamma(\mathcal{G})$.

Proposition (Raum - F.)

Let $\mathcal{G}$ be a groupoid whose orbit space is (T_0). Let $x \in \mathcal{G}^{(0)}$ be a unit, write $G = \mathcal{G}|_x$ for the isotropy group at x and denote by G_0 the associated group with zero. Then G_0 is a quotient of a corner of $\Gamma(\mathcal{G})$. Vice versa, if $\mathcal{G}$ is any ample Hausdorff groupoid and G is a group such that G_0 is a quotient of a corner of $\Gamma(\mathcal{G})$, then G is a quotient of a point stabiliser of $\mathcal{G}$.

Main theorem

Putting together both propositions, we got the following characterization.

Theorem (Raum - F.)

Let $\mathcal{G}$ be a second countable, ample Hausdorff groupoid. Then $\mathcal{G}$ is type I if and only if the following two conditions are satisfied.

1. No corner of $\Gamma(\mathcal{G})$ has a non virtually abelian group quotient, and
2. $\Gamma(\mathcal{G})$ does not have an infinite, monoidal and 0-simplifying subquotient.

Boolean inverse completions

- Let S be an inverse semigroup and E its idempotents.
- Consider the set $\widehat{E}$ of multiplicative non-zero maps $E \to \{0, 1\}$.
- Let $D_s := \{\chi \in \widehat{E} \mid \chi(s^*s) = 1\} \subseteq \widehat{E}$.
- $s \in S$ sends D_s to D_{s^*} via the action $s \cdot \chi(e) = \chi(s^*es)$.
- It gives a partial action of S on $\widehat{E}$. Call $\mathcal{G}(S)$ the transformation groupoid associated to this action.
- $B(S) := \Gamma(\mathcal{G}(S))$ is the *boolean inverse completion* of S.

Fact

This construction is left adjoint to the forgetful functor from boolean inverse semigroups to inverse semigroups with 0.

Characterization for inverse semigroups

Realizing quotients of corners of $B(S)$ as subquotients of S, we obtained the following result.

Theorem (Raum - F.)

Let S be an inverse semigroup. S is type I if and only if the following two conditions are satisfied

1. S does not have any non virtually abelian group subquotient, and
2. $B(S)$ does not have an infinite, monoidal, 0-simplifying subquotient.