

A dynamical approach to Hecke operator algebras

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Let me first remind you of
some things Adam told us about!

Coxeter groups

Definition

A *Coxeter group* W is a group of the form

$$W := \langle S \mid (st)^{m_{st}} = e \rangle$$

where S is a (for the purposes of this talk) finite set and $m_{st} \in \{1, 2, \dots, \infty\}$ with $m_{ss} = 1$, $m_{st} = m_{ts}$ and $m_{st} \geq 2$ for all $s \neq t \in S$. The pair (W, S) is called a *Coxeter system*.

It is *right-angled* if $m_{st} \in \{2, \infty\}$ for all $s \neq t \in S$.

It is *irreducible* if (W, S) does not decompose as a non-trivial direct product of the form $W = \langle T_1 \rangle \times \langle T_2 \rangle$ where $T_1, T_2 \subseteq S$.

Hecke operator algebras

Definition

Let (W, S) be a Coxeter system, $q > 0$ and define $T_s^{(q)} \in \mathcal{B}(\ell^2(W))$, $s \in S$ by

$$T_s^{(q)} \delta_w := \begin{cases} \delta_{sw} & , \text{ if } |sw| > |w| \\ \delta_{sw} + p_q \delta_w & , \text{ if } |sw| < |w| \end{cases} ,$$

where $p_q := (q-1)/\sqrt{q}$. Then $C_{r,q}^*(W) := C^*(\{T_s^{(q)} \mid s \in S\})$ is called the corresponding *Hecke C^* -algebra* and $\mathcal{N}_q(W) := (C_{r,q}^*(W))'' \subseteq \mathcal{B}(\ell^2(W))$ is the corresponding *Hecke-von Neumann algebra*.

Think of Hecke C^* -algebras (resp. Hecke-von Neumann algebras) as “deformations” of $C_r^*(W)$ (resp. of $\mathcal{L}(W)$).

Two questions

What is the structure of these operator algebras? More concretely:

- 1 For which Coxeter systems (W, S) and $q > 0$ is $C_{r,q}^*(W)$ simple?
- 2 For which Coxeter systems (W, S) and $q > 0$ is $\mathcal{N}_q(W)$ a factor?

Remark: It suffices to answer these questions for irreducible Coxeter systems.

A few results into that direction:

Theorem (Fendler, de la Harpe, Cornulier 2003-2012)

Let (W, S) be an irreducible Coxeter system. Then $C_r^*(W)$ is simple if and only if W is not virtually abelian.

Theorem (Caspers-K.-Larsen 2019)

Let (W, S) be an irreducible Coxeter system, $q > 0$ and let ρ be the radius of convergence of $\sum_{w \in W} z^{|w|}$.

- If $q \notin (\rho, \rho^{-1})$, then $C_{r,q}^*(W)$ is not simple. In particular, if W is virtually abelian, then $C_{r,q}^*(W)$ is never simple.
- Let (W, S) be right-angled and $|S| \geq 3$. Then, if q is close enough to 1, $C_{r,q}^*(W)$ is simple.

Theorem (Garncarek 2015, Raum-Skalski 2020)

Let (W, S) be a right-angled irreducible Coxeter system with $|S| \geq 3$ and $q > 0$. Then $\mathcal{N}_q(W)$ is a factor if and only if $q \in [\rho, \rho^{-1}]$ where ρ is the radius of convergence of $\sum_{w \in W} z^{|w|}$.

How to make progress?

In theory of group C^* -algebras and group
-von Neumann algebras, studying certain
boundaries turned out to be very useful.

So let's build boundaries associated with Coxeter systems
that relate to our Hecke C^* -algebras!

A different approach

Let (W, S) be a Coxeter system. For $w \in W$ let $P_w \in \mathcal{B}(\ell^2(W))$ be the orthogonal projection onto

$$\overline{\text{Span}\{\delta_v \mid v \in W \text{ with } |w^{-1}v| = |v| - |w|\}} \subseteq \ell^2(W).$$

Further, let $\mathcal{K} \subseteq \mathcal{B}(\ell^2(W))$ be the compact operators and let $\pi: \mathcal{B}(\ell^2(W)) \rightarrow \mathcal{B}(\ell^2(W))/\mathcal{K}$ be the quotient map.

Definition

Set $\mathcal{D}(W, S) := C^*(\{P_w \mid w \in W\})$ and call $\overline{(W, S)} := \text{Spec}(\mathcal{D}(W, S))$ the *compactification* of (W, S) . We call $\partial(W, S) := \text{Spec}(\pi(\mathcal{D}(W, S)))$ the *boundary* of (W, S) . (These spaces have been introduced earlier by Caprace-Lécureux and Lam-Thomas.)

$\overline{(W, S)}$ and $\partial(W, S)$ are metrizable compact spaces and $W \subseteq \overline{(W, S)}$ is dense and open.

The connection to Hecke C^* -algebras

The group W acts on itself by left multiplication. This action induces (continuous) actions $W \curvearrowright \overline{(W, S)}$ and $W \curvearrowright \partial(W, S)$. In particular, we can build the corresponding crossed products $C(\overline{(W, S)}) \rtimes_{red} W$ and $C(\partial(W, S)) \rtimes_{red} W$.

Embeddings:

- For every $q > 0$ there exists a natural embedding

$$C_{r,q}^*(W) \hookrightarrow C(\overline{(W, S)}) \rtimes_{red} W.$$

- For every $q \in [\rho, \rho^{-1}]$ there exists a natural embedding

$$C_{r,q}^*(W) \hookrightarrow C(\partial(W, S)) \rtimes_{red} W.$$

Questions: What are the properties of the actions $W \curvearrowright \overline{(W, S)}$ and $W \curvearrowright \partial(W, S)$? What can they tell us about the Hecke operator algebras?

Boundary actions

The actions $W \curvearrowright \overline{(W, S)}$ and $W \curvearrowright \partial(W, S)$ have a number of interesting properties. But let's restrict to a particular one:

Definition

Let G be a discrete group and X a compact Hausdorff space. An action $G \curvearrowright X$ is called...

- *minimal* if $G.x := \{g.x \mid g \in G\} \subseteq X$ is dense for every $x \in X$;
- *strongly proximal* if for every probability measure ν on X the weak- $*$ -closure of the G -orbit $G.\nu$ contains a point mass δ_x for some $x \in X$.

X is called a *G -boundary* if the action $G \curvearrowright X$ is both minimal and strongly proximal.

Boundary actions play an important role in Kalantar-Kennedy's approach to the simplicity question of group C^* -algebras.

In some cases we can characterize when the action of a Coxeter group W on $\partial(W, S)$ is a boundary action. For instance:

Theorem

Let (W, S) be a right-angled, irreducible Coxeter system with $|S| \geq 2$. Then $\partial(W, S)$ is a W -boundary if and only if $|S| \geq 3$.

Further, in that case the action $W \curvearrowright \partial(W, S)$ is *topologically free* in the sense that for every $w \in W \setminus \{e\}$ the set $\{z \in \partial(W, S) \mid w.z = z\}$ has no inner points.

In particular, if (W, S) is right-angled with $|S| \geq 3$, the reduced crossed product C^* -algebra $C(\partial(W, S)) \rtimes_{red} W$ is simple.

Simplicity of right-angled Hecke C^* -algebras

We can now approach the simplicity question for right-angled Hecke C^* -algebras.

Theorem

Let (W, S) be a right-angled, irreducible Coxeter system with $|S| \geq 3$ and $q > 0$. Then $C_{r,q}^*(W)$ is simple if and only if $q \in (\rho, \rho^{-1})$.

The proof goes back to an idea of Haagerup.

Proof ^a (in the case $q = 1$)

^a "... but this margin is too small to contain."

Let $I \neq C_{r,1}^*(W) \subseteq C(\partial(W, S)) \rtimes_{red} W$ be an ideal in $C_{r,1}^*(W)$ and let ϕ be a state on $C(\partial(W, S)) \rtimes_{red} W$ that vanishes on I .

Since $\overline{W \curvearrowright \partial(W, S)}$ is strongly proximal, there exists $x \in \partial(W, S)$ with $\delta_x \in W \cdot (\phi|_{C(\partial(W, S))})$. So we find a sequence $(w_i)_{i \in \mathbb{N}} \subseteq W$ such that $w_i \cdot \phi \xrightarrow{\text{weak-}^*} \psi$ for some state ψ with $\psi|_{C(\partial(W, S))} = \delta_x$ and $\psi|_I \equiv 0$.

But then, for every $a \in I$, $f, g \in C(\partial(W, S))$ and $v, w \in W$,

$$\psi((f \lambda_v) a (\lambda_w g)) = f(x) \psi(\lambda_v a \lambda_w) g(x) = 0.$$

That implies that $\psi|_J \equiv 0$ where $J \triangleleft C(\partial(W, S)) \rtimes_{red} W$ is the ideal generated by I . But $\psi \neq 0$, so $J \neq C(\partial(W, S)) \rtimes_{red} W$. Since $C(\partial(W, S)) \rtimes_{red} W$ is simple we get that $J = 0$ and hence $I = 0$. (■)

The general proof is more complicated, but the basic idea is the same.

Some questions

- Is the action $W \curvearrowright \partial(W, S)$ always minimal?
- Can we characterize when the action $W \curvearrowright \partial(W, S)$ is strongly proximal? (Guess: It is strongly proximal if and only if W is not virtually abelian.)
- Can we characterize when the action $W \curvearrowright \partial(W, S)$ is topologically free? (Guess: It is topologically free if and only if W is not virtually abelian.)
- What (else) can the boundary $\partial(W, S)$ tell us about the ideal structure of Hecke C^* -algebras?

Thank you!