Uniformly recurrent subgroups and applications

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$$O(K; U_1, \ldots, U_n) = \{H \in \mathrm{Sub}(G) \,:\, H \cap K = \emptyset;\ H \cap U_i \neq \emptyset \text{ for all } i\},$$

where $K \subset G$ *is compact and* $U_1, \ldots, U_n \subset G$ *are open, form a basis for the Chabauty topology on* $\mathrm{Sub}(G)$.

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For G discrete: (H_n) converges to H if and only if:

- for every $h \in H$, $h \in H_n$ for n large enough;
- if $g \in H_n$ for infinitely many n, then $g \in H$.

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Fact

$\mathrm{Sub}(G)$ *is a compact space.*

There is an action $G \curvearrowright \mathrm{Sub}(G)$, $(g, H) \mapsto gHg^{-1}$. $\mathcal{O}_H$ will denote the orbit of H, ie the conjugacy class of H.

URSs

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Recall: Invariant random subgroups (IRS) are G-invariant probability measures on $\mathrm{Sub}(G)$. Very much studied.

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We call $\{\{1\}\}$ *the trivial URS.*

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If $\mathcal{H}$ is a non-trivial URS, then every $H \in \mathcal{H}$ is confined because $\overline{\mathcal{O}_H} = \mathcal{H}$.
Conversely if H is confined then $\overline{\mathcal{O}_H}$ contains a non-trivial URS.

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Definition

A minimal action of G on X is topologically free if there exists a dense G_δ set of points $x \in X$ such that $G_x = \{1\}$. This is equivalent to saying that the URS associated to the action of G on X is the trivial URS.

Works on URSs

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- For G a simple locally finite group, confined subgroups of G are closely related to ideals in the group algebra $\mathfrak{K}G$ (93–03: Zalesskii, Sehgal, Hartley, Leinen, Puglisi).

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- Relate other actions on compact spaces $G \curvearrowright Y$ with $G \curvearrowright X$.
- Applications about problems on G a priori not related to X.

Branch groups

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(Grigorchuk, Wilson, Bartholdi, Nekrashevych, Sunik,...)

Branch groups

T is a locally finite rooted tree. The set of vertices of T at distance n from the root form the n-th level of T, denoted by $L(n)$. For $v \in T$, T_v is the subtree of T of vertices below v. Let $G \leq \mathrm{Aut}(T)$.

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The rigid stabilizer $\mathrm{Rist}_G(v)$ a vertex v is the pointwise fixator of the complement of T_v in G. For $n \geq 1$, we denote by $\mathrm{Rist}_G(n) \simeq \prod_{v \in L(n)} \mathrm{Rist}_G(v)$ the subgroup generated by $\mathrm{Rist}_G(v)$ when v ranges over $L(n)$.

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Remark

This falls into the above general setting with $G \curvearrowright X = \partial T$.

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Remark

Specific to URSs (not true for all confined subgroups).

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What follows uses this theorem.

Non topologically free actions of branch groups

Theorem

Let G be a branch group, and X a compact G-space on which the action is faithful, minimal and not topologically free. Then the G-action on X factors onto a non-trivial profinite G-space Y.

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Corollary

Let X a compact G-space on which the action is faithful, minimal and proximal. Then the action is topologically free.

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Theorem (Frisch–Tamuz–Vahidi Ferdowsi + Glasner–Tsankov–Weiss–Zucker)

Every countable ICC group admits a topologically free minimal and proximal action.

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- Does not depend on S up to equivalence relation.
- If $E = G/H$, this is the growth of the Schreier graph of H.

Orbital growth

Let $G = \langle S \rangle$ be a finitely generated group. Let E be a G-set. The **growth of the action** of G on E is

$$\mathrm{vol}_{G,E}(n) = \sup_{x \in E} |S^n x|.$$

- Does not depend on S up to equivalence relation.
- If $E = G/H$, this is the growth of the Schreier graph of H.
- $\mathrm{vol}_{G,E}(n) \leq \mathrm{vol}_G(n)$.

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Question

Given a f.g. group G, can G admit actions with slow (linear, quadratic, polynomial, subexponential...) growth ?

Orbital growth of branch groups

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Theorem

Let $G \leq \mathrm{Aut}(T)$ be a finitely generated branch group. Then for every G-set E on which G acts faithfully, we have

$$\mathrm{vol}_{G,E} \succeq \mathrm{vol}_{G,\partial T}.$$

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Remark

The proof also relies on finite generation of normal subgroups of f.g. branch groups (Francoeur).