Schützenberger graphs and property A, or how to see exactness of the reduced semigroup C*-algebra

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C*-algebras and geometry of groups and semigroups

University of Oslo

Outline

(1) Inverse semigroups

(2) Schützenberger graphs and right invariance

(3) Property A, and its relation with exactness

(4) *Having finite local structure*

1. Inverse semigroups

Inverse semigroups and Wagner-Preston

Definition

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- $E(S) = \{e \in S \mid e^2 = e\} = \{s^*s \mid s \in S\}$ is commutative
- $D_{s^*s} := \{x \in S \mid x = s^*sx\} = s^*s \cdot S$ is the *domain of* s^*s
- $s \colon D_{s^*s} \to D_{ss^*}$, where $x \mapsto sx$ is a bijection

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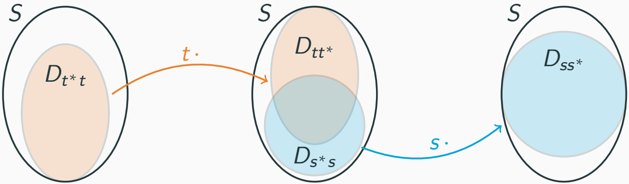

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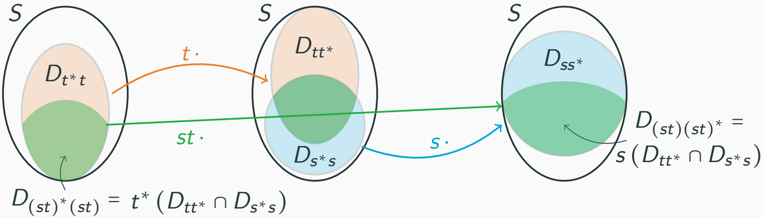

Group coarse geometry

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

- Graph $\rightsquigarrow \operatorname{Cay}(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \left\{ \left(x, g_i^{\pm 1} x \right) \mid x \in G \text{ and } i = 1, \ldots, n \right\}$.

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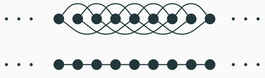

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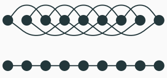

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Proposition (classical)

The large scale geometry of the Cayley graph of G
does not depend on the generators

Goal: reproduce these constructions for inverse semigroups

2. Schützenberger graphs and right invariance

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces:

$$\text{if } x \in D_{s^*s} = \{y \in S \mid s^*sy = y\} \text{ then } (sx)^*(sx) = x^*x$$

and hence $x \mathcal{L} sx$ (the converse also holds)

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Green's relations:

- $x\mathcal{L}y$ if $x^*x = y^*y$
- $x\mathcal{R}y$ if $xx^* = yy^*$
- $\mathcal{H} = \mathcal{L}$ and $\mathcal{R}$
- $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$

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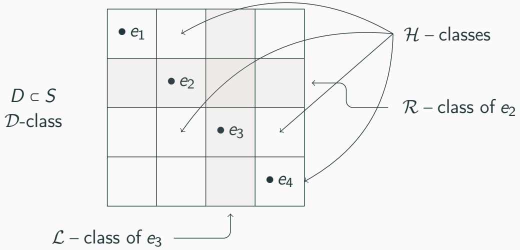

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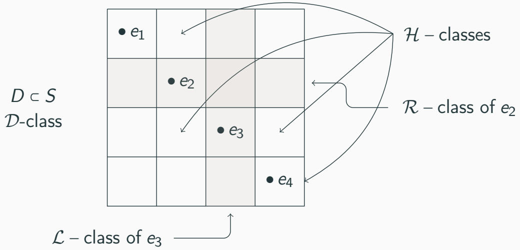

Remark - need of extended metrics

Good distances $d: S \times S \to [0, \infty]$ satisfy that

$$x^*x = y^*y \Leftrightarrow d(x, y) < \infty$$

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
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Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

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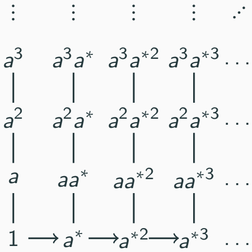

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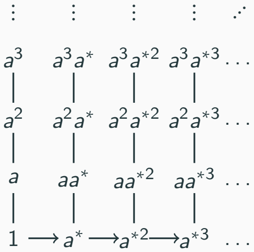

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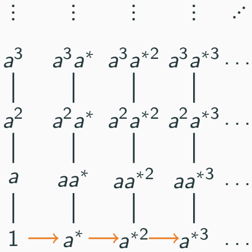

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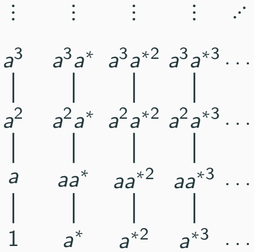

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$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 & a^3a^* & a^3a^{*2} & a^3a^{*3} & \dots \\
| & | & | & | & \\
a^2 & a^2a^* & a^2a^{*2} & a^2a^{*3} & \dots \\
| & | & | & | & \\
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| & | & | & | & \\
1 & a^* & a^{*2} & a^{*3} & \dots
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Schützenberger graphs II: right invariance

Remark: not all graphs are group Cayley graphs. However:

Theorem (Stephen - 1990)

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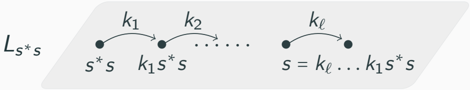

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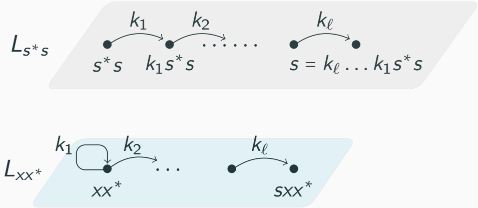

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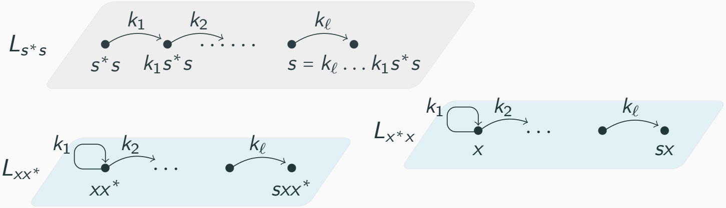

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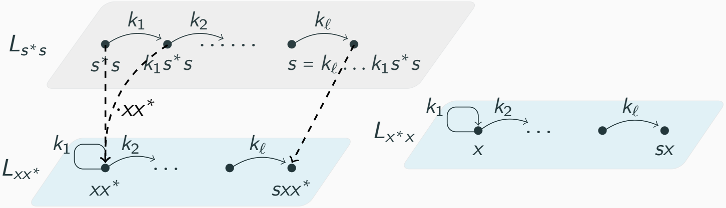

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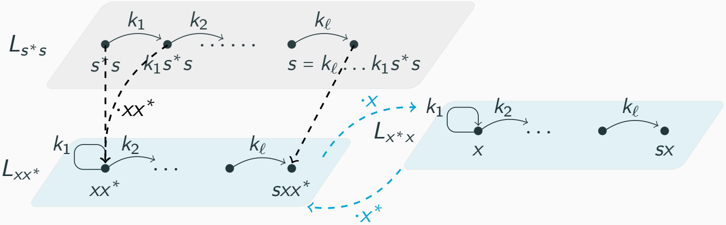

3. Property A, and its relation with exactness

Schützenberger graphs and property A

Definition (Yu - 1999)

(X,d) has *property A* if for every $r, \varepsilon > 0$ there is

$$\xi \colon X \to \ell^1(X)_1^+ \text{ and } c > 0 \text{ such that } \operatorname{supp}(\xi_x) \subset B_c(x) \text{ and}$$

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Theorem (Ozawa - 2000)

Let G be a countable group. The following are equivalent:

(1) G has property A.
(2) $\ell^\infty(G)\rtimes_r G$ is a nuclear C*-algebra.
(3) $C_r^*(G)$ is an exact C*-algebra.

Property A, nuclearity and exactness for inverse semigroups

Left regular representation: $V\colon S \to \mathcal{B}\left(\ell^2 S\right)$, where

$$V_s \delta_x = \begin{cases} \delta_{sx}, & \text{if } x \in D_{s^* s} \\ 0 & \text{otherwise} \end{cases}$$

Reduced C*-algebra: $C_r^*(S) := C^*\left(\{V_s\}_{s \in S}\right) \subset \mathcal{B}\left(\ell^2 S\right)$

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Let $S = \langle K \rangle$ be an inverse semigroup.

If Λ_S has property A (as a graph), then $C_r^*(S)$ is exact.

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The converse holds in *some* cases.

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Let $S = \langle K \rangle$ be an inverse semigroup.

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- The converse holds in *some* cases.

Proof: direct argument from A $\rightsquigarrow$ $C_r^*\left(S\right)$ exact

Question: when?

Answer: when the local structure of Λ_S is not too complex

4. Having finitely complex local structures

Finite labeling I: local structure

Note: Λ_S = vertices + edges, and edges represent (x, sx)

Finite labeling I: local structure

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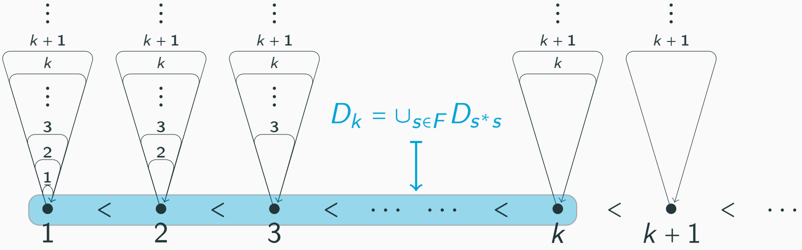

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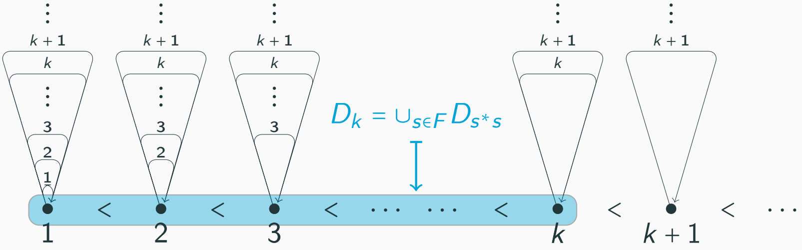

Key: for large $r \geq 0 \nexists F \Subset S$ labeling the paths in Λ_S of length r

Finite labeling II: a picture to *top* the explanation

Definition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. We say (S, K) admits a *finite labeling* if

for any $r \geq 0$ there is $F \Subset S$ such that if $d(s^*s, s) \leq r$

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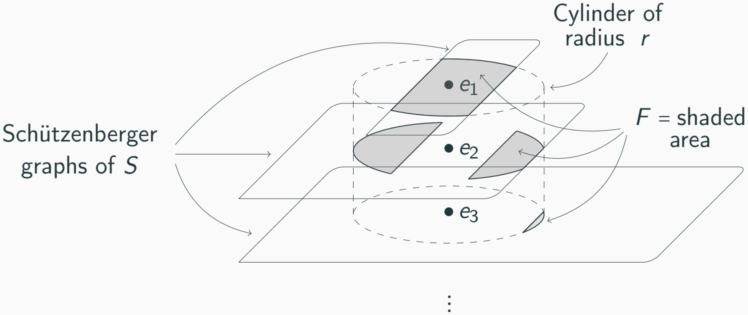

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Theorem (Lledó, M. - 2021)

Suppose that (S, K) admits a finite labeling.

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Thank you for your attention. Questions?

Paterson's universal groupoid and its amenability

Theorem (Paterson - 1999)

Given S there is $G_{\mathcal{U}}(S) = \{\text{filters of } E(S)\} \rtimes_\theta S$ such that

$$C_r^*(S) = C_r^*(G_{\mathcal{U}}(S)) \text{ and } C^*(S) = C^*(G_{\mathcal{U}}(S))$$

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Recall: for arbitrary (discrete) groups:

$$G \text{ is amenable} \Leftrightarrow C_r^*(G) \text{ is nuclear} \Leftrightarrow C_r^*(G) = C^*(G)$$

And this has become a driving force of groupoid amenability:

Theorem (Anantharaman-Delaroche and Renault)

An (étale) groupoid $\mathcal{G}$ is amenable $\Leftrightarrow C_r^*(\mathcal{G})$ is nuclear.

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Question:

- Relation between $G_{\mathcal{U}}(S)$ amenable and S amenable?
- How does $G_{\mathcal{U}}(S)$ enter the picture?

However, property A of $\Lambda_S \rightsquigarrow$ exactness of $C_r^*(S)$:

Proposition (Lledó, M. - 2021)

Let $S = \langle K \rangle$ with (S, K) admitting a finite labeling.

If $G_{\mathcal{U}}(S)$ is amenable then Λ_S has property A.

Groupoid amenability and property A

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Proof: $G_{\mathcal{U}}(S)$ amenable $\Leftrightarrow C_r^*(S) = C_r^*(G_{\mathcal{U}}(S))$ nuclear

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Remarks:

- Alternate argument via the definitions involved $\rightsquigarrow$ not C*
- Is <u>not</u> an equivalence, as $\mathbb{F}_2$ has prop. A and is not amenable

E-unitary semigroups and relation with property A

Recall: $G(S) = S/\sigma$, where $s\sigma t$ iff $se = te$ for some $e \in E(S)$

- Known as the *maximal homomorphic image of S*
- $G(S)$ is always a group, and let $\sigma\colon S \to G(S)$ the quotient map

Theorem (Duncan and Namioka - 1978)

S is amenable if, and only if, $G(S)$ is amenable.

However: $G(S)$ loses information of $S \rightsquigarrow$ might even $G(S) = \{1\}$

Definition (classical)

S is *E-unitary* if $\sigma\colon S \to G(S)$ is injective in every $\mathcal{L}$-class.

Theorem (Anantharaman-Delaroche - 2016)

Let S be E-unitary. $C_r^*(S)$ is exact $\Leftrightarrow$ $G(S)$ is exact.

E-unitary semigroups and relation with property A

Goal: prove Anantharaman-Delaroche's result geometrically

Theorem (Lledó, M. - 2021)

Let $S = \langle K \rangle$ be E-unitary and (S, K) admit a finite labeling.

Λ_S has property A $\Leftrightarrow$ $G(S)$ has property A.

(Recall that, by Ozawa, property A = exactness for groups)

Proof $\Rightarrow$**:** given $\xi\colon S \to \ell^1(S)^1_+$ for S let

$$\zeta\colon G(S) \to \ell^1(G(S)), \text{ where } \zeta_{\sigma(s)}(\sigma(t)) := \lim_{n\to\omega} \xi_{se_n}(te_n)$$

for some $e_1 \geq \cdots \geq e_n \geq \cdots \in E(S)$ is *eventually below everything*

- Locally $G(S)$ is Λ_{L_e} for e sufficiently small
- Hence, the same approx. for Λ_S does the trick for $G(S)$

Proof $\Leftarrow$**:** only known via C*-arguments

Higson-Lafforgue-Skandalis and Willett's example

Example: box space without property A

- $\mathbb{F}_2 = \langle a, b \mid - \rangle$ free non-abelian group on 2 elements
- Let $\{N_k\}_{k\in\mathbb{N}}$ be a descending sequence of normal subgroups of finite index such that $\cap_{k\in\mathbb{N}} N_k = \{1\}$
- $S := \sqcup_{k\in\mathbb{N}} \mathbb{F}_2/N_k$, where $[g]_i \cdot [h]_j := [gh]_{\min\{i,j\}}$.

Remark:

(1) S is amenable $\rightsquigarrow$ as it has a 0

(2) S does not have property A $\rightsquigarrow$ $\mathcal{L}$-classes are *complicated*

(3) S does not admit a finite labeling $\rightsquigarrow$ same as $(\mathbb{N}, \min)$