

On property (T) for $\text{Aut}(F_n)$

Piotr Nowak

Institute of Mathematics
Polish Academy of Sciences
Warsaw

Property (T)

G - group generated by a finite set S

Definition (Kazhdan 1966)

G has property (T) if there is $\kappa = \kappa(G, S) > 0$ such that

$$\sup_{s \in S} \|v - \pi_s v\| \geq \kappa \|v\|$$

for every unitary representation without invariant vectors.

Kazhdan constant = optimal $\kappa(G, S)$

\iff the trivial rep is an isolated point in the unitary dual of G with the Fell topology

Property (T)

Finite groups have (T)

Classical examples of infinite groups with (T):

- higher rank simple Lie groups and their lattices $SL_n(\mathbb{R})$, $SL_n(\mathbb{Z})$, $n \geq 3$ (Kazhdan, 1966)
- automorphism groups of certain buildings, e.g. \tilde{A}_2 (Cartwright-Młotkowski-Steger 1996, Pansu, Żuk, Ballmann-Świątkowski, 1997-98)
- certain random hyperbolic groups in the triangular and Gromov model (Żuk 2003)

Examples without (T): amenable, free, groups with infinite abelianization, groups acting unboundedly on finite-dim. CAT(0) cube complexes (Niblo-Reeves)

Why is property (T) interesting?

Examples of applications:

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- counterexamples to Baum-Connes type conjectures (Higson-Lafforgue-Skandalis 2003): K -theory classes represented by Kazhdan-type projections do not lie in the image of appropriate Baum-Connes assembly map

Question

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For small values of n the answer is negative:

- $\text{Aut}(F_2)$ maps onto $\text{Out}(F_2) \simeq \text{GL}_2(\mathbb{Z})$
- $\text{Aut}(F_3)$ maps onto \mathbb{Z} (McCool 1989)
and virtually onto F_2 (Grunewald-Lubotzky 2006)

\implies no (T)

Ozawa's characterization

Laplacian in the real group ring $\mathbb{R}G$:

$$\Delta = |S| - \sum_{s \in S} s \quad \in \mathbb{R}G$$

Theorem (Ozawa, 2014)

(G, S) has property (T) iff for some $\lambda > 0$ and a finite collection of $\xi_i \in \mathbb{R}G$

$$\Delta^2 - \lambda\Delta = \sum_{i=1}^n \xi_i^* \xi_i$$

This is a finite-dim condition - perhaps possible to solve with computer assistance.

Property (T) quantified

If the equation is satisfied:

$$\Delta^2 - \lambda\Delta = \sum_{i=1}^n \xi_i^* \xi_i$$

then relation to Kazhdan constants:

$$\sqrt{\frac{2\lambda}{|S|}} \leq \kappa(G, S)$$

we can also define the following notion

Definition

Kazhdan radius of (G, S) = smallest $r > 0$ such that $\text{supp } \xi_i \subseteq B(e, r)$ for all ξ_i above.

Implementation for $SL_n(\mathbb{Z})$

Implementation of this strategy pioneered by Netzer-Thom (2015):

a new, computer-assisted proof of (T) for $SL_3(\mathbb{Z})$
(generators: elementary matrices)

Improvement of Kazhdan constant:

$$\simeq 1/1800 \quad \rightarrow \quad \simeq 1/6$$

Later also improved Kazhdan constants for $SL_n(\mathbb{Z})$ for

$n = 3, 4$ (Fujiwara-Kabaya)

$n = 3, 4, 5$ (Kaluba-N.)

Setup: $\text{SAut}(F_n)$

Subgroup of $\text{Aut}(F_n)$, generated by Nielsen transformations:

$$R_{ij}^{\pm}(s_k) = \begin{cases} s_k s_j^{\pm 1} & \text{if } k = i \\ s_k & \text{oth.} \end{cases}, \quad L_{ij}^{\pm}(s_k) = \begin{cases} s_j^{\pm 1} s_k & \text{if } k = i \\ s_k & \text{oth.} \end{cases}$$

Equivalently,

$$\text{SAut}(F_n) = ab^{-1}(\text{SL}_n(\mathbb{Z}))$$

under the map $\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})$ induced by the abelianization $F_n \rightarrow \mathbb{Z}^n$.

$\text{SAut}(F_n)$ has index 2 in $\text{Aut}(F_n)$

Main results

Theorem (Kaluba - N. - Ozawa)

SAut(F_5) has property (T) with Kazhdan constant

$$0.18 \leq \kappa(\text{SAut}(F_5))$$

Theorem (Kaluba - Kielak - N.)

SAut(F_n) has property (T) for $n \geq 6$ with Kazhdan radius 2 and Kazhdan constant

$$\sqrt{\frac{0.138(n-2)}{6(n^2-n)}} \leq \kappa(\text{SAut}(F_n)).$$

Certifying positivity via semidefinite programming

$\eta \in \mathbb{R}G$ positive if η is a sum of squares:

$$\eta = \sum_{i=1}^k \xi_i^* \xi_i$$

where $\text{supp } \xi_i \subseteq E$ - finite subset of G . (here always $E = B(e, 2)$)

\iff

there is a positive definite $E \times E$ matrix P such that for $\mathbf{b} = [g_1, \dots, g_n]_{g_i \in E}$

$$\eta = \mathbf{b}P\mathbf{b}^T = \mathbf{b}Q Q^T \mathbf{b}^T = (\mathbf{b}Q)(\mathbf{b}Q)^T$$

i -th column of Q = coefficients of ξ_i

Using the solver

To check if η is positive we can use a semidefinite solver to perform convex optimization over positive definite matrices:

find $P \in \mathbb{M}_{E \times E}$ such that:
 $\eta = \mathbf{b}P\mathbf{b}^T$
positive semi-definite

Assume now that a computer has found a solution P - we obtain

$$\eta \simeq \sum \xi_i^* \xi_i$$

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However this can be improved if the error is small in a certain sense and η belongs to the augmentation ideal IG

Lemma (Ozawa, Netzer-Thom)

Δ is an order unit in IG : if $\eta = \eta^* \in IG$ then

$$\eta + R\Delta = \sum \xi_j^* \xi_j$$

for all sufficiently large $R \geq 0$.

Moreover, $R = 2^{2r-2} \|\eta\|_1$ is sufficient, where $\text{supp } \eta \subseteq B(e, 2^r)$.

With this in mind we can try to improve our search of P to show that η is “strictly positive”:

maximize $\lambda \geq 0$ under the conditions

$$\eta - \lambda \Delta = \mathbf{b}P\mathbf{b}^T$$

$P \in \mathbb{M}_{E \times E}$ positive semi-definite

Assume now that numerically on $B(e, 2^r)$

$$\eta - \lambda\Delta \simeq \mathbf{b}P\mathbf{b}^T$$

and $P = QQ^T$.

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Correct Q to \bar{Q} , where columns of Q sum up to 0.

$$\left\| \eta - \lambda\Delta - \mathbf{b}\bar{Q}\bar{Q}^T\mathbf{b}^T \right\|_1 \leq \varepsilon$$

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$$\left\| \eta - \lambda\Delta - \mathbf{b}\bar{Q}\bar{Q}^T\mathbf{b}^T \right\|_1 \leq \varepsilon$$

Then for $R = \varepsilon 2^{2r-2}$ or larger:

$$\eta - \underbrace{\lambda\Delta + R\Delta}_{(\lambda-R)\Delta} = \underbrace{\mathbf{b}\bar{Q}\bar{Q}^T\mathbf{b}^T}_{\sum \eta_i^* \eta_i} + \underbrace{(\eta - \lambda\Delta - \mathbf{b}\bar{Q}\bar{Q}^T\mathbf{b}^T + R\Delta)}_{\geq 0 \text{ by lemma}}$$

If $\lambda - R > 0$ then $\eta - (\lambda - R)\Delta \geq 0$ and in particular, $\xi \geq 0$.

Important:

The ℓ_1 -norm is computed in interval arithmetic.

This gives a mathematically rigorous proof of positivity of η .

To prove (T): apply to $\eta = \Delta^2 - \lambda\Delta$ and maximize for $\lambda > 0$

We want to use this strategy in $\text{SAut}(F_5)$.

Problem #1:

$$|B(e, 2)| = 4\,641$$

P has 10 771 761 variables - too large for a solver to handle

Symmetrization

We reduce the number of variables via symmetries

$\Gamma = \mathbb{Z}_2 \wr \text{Sym}_n$ acts on generators of F_n by inversions and permutations

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Γ also acts on $\text{Aut}(F_5)$ by conjugation

preserves $\text{SAut}(F_n)$ and $\Delta^2 - \lambda\Delta$

There is an induced action on \mathbb{M}_E - $E \times E$ matrices

Lemma

If there is a solution P then there is also a Γ -invariant solution P .

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Problem #2: solvers do not work with Γ -invariant matrices...

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ρ_E = permutation representation of Γ on $\ell_2(E)$

$$\rho_E = \bigoplus_{\pi \in \hat{\Gamma}} m_\pi \pi$$

where m_π = multiplicity of π

Theorem (Wedderburn decomposition)

$$C^*(\rho_E) \simeq \bigoplus_{\pi \in \hat{\Gamma}} M_{\dim \pi} \otimes 1_{m_\pi}$$

Thus: $M_E^\Gamma \simeq (C^*(\rho_E))' \simeq \bigoplus_{\pi \in \hat{\Gamma}} 1_{\dim \pi} \otimes M_{m_\pi}$

We need this isomorphism to be explicit - this can be done using a system of minimal projections provided by representation theory

Theorem (Kaluba-N.-Ozawa)

$\text{SAut}(F_5)$ has (T) with Kazhdan constant ≥ 0.18027 .

Proof.

Find sum of squares decomposition for $\Delta^2 - \lambda\Delta$
on the ball of radius 2

Data from solver: P and $\lambda = 1.3$

$$8.30 \cdot 10^{-6} \leq \left\| \Delta^2 - \lambda\Delta - \sum \xi_i^* \xi_i \right\|_1 \leq 8.41 \cdot 10^{-6}$$

$\implies R = 8.41 \cdot 10^{-6} \cdot 2^{4-2}$ suffices

$$\lambda - R = 1.2999 > 0$$

Property (T) for $\text{Aut}(F_n)$, $n \geq 6$

G_n with generating set S_n will denote either one of the families

- $\text{SAut}(F_n)$ generated by Nielsen transformations $R_{i,j}^\pm, L_{i,j}^\pm$
- $\text{SL}_n(\mathbb{Z})$ generated by elementary matrices $E_{i,j}^\pm$

G_n form a tower via the inclusions of

$$F_n \subseteq F_{n+1} \quad \text{and} \quad \mathbb{Z}_n \subseteq \mathbb{Z}_{n+1}$$

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\mathcal{C}_n - $(n - 1)$ -simplex on $\{1, \dots, n\}$

E_n set of edges of $\mathcal{C}_n =$ unoriented pairs $e = \{i, j\}$

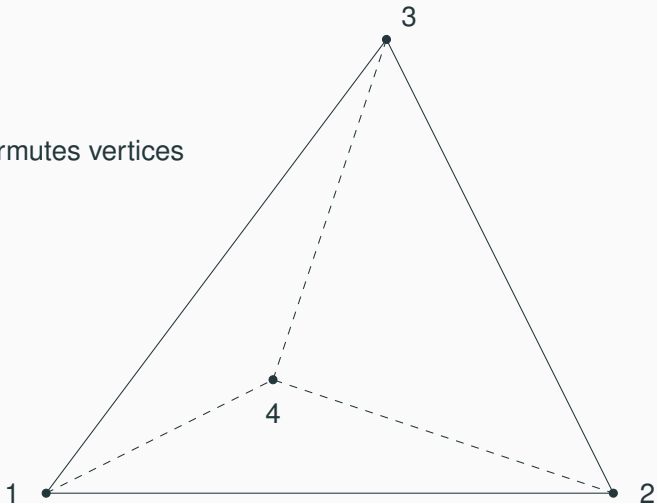
Alt_n acts on edges: $\sigma(e) = \sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\}$

Map

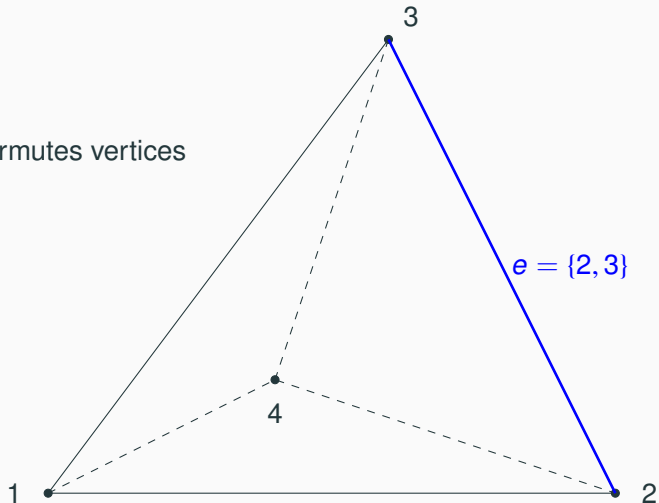
$$l_n : \mathcal{S}_n \rightarrow \mathcal{C}_n,$$

$$R_{ij}^\pm, L_{ij}^\pm \mapsto \{i, j\}, \quad E_{ij}^\pm \mapsto \{i, j\}$$

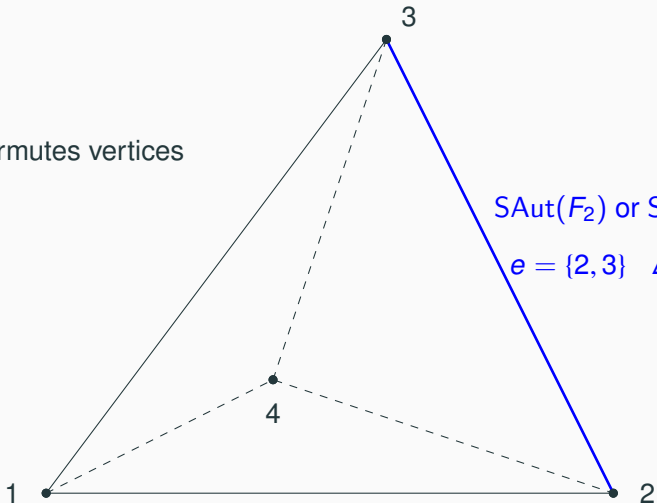
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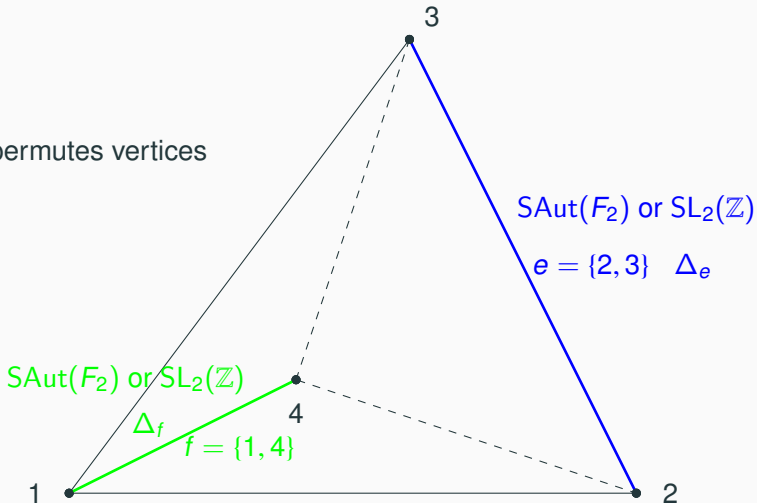
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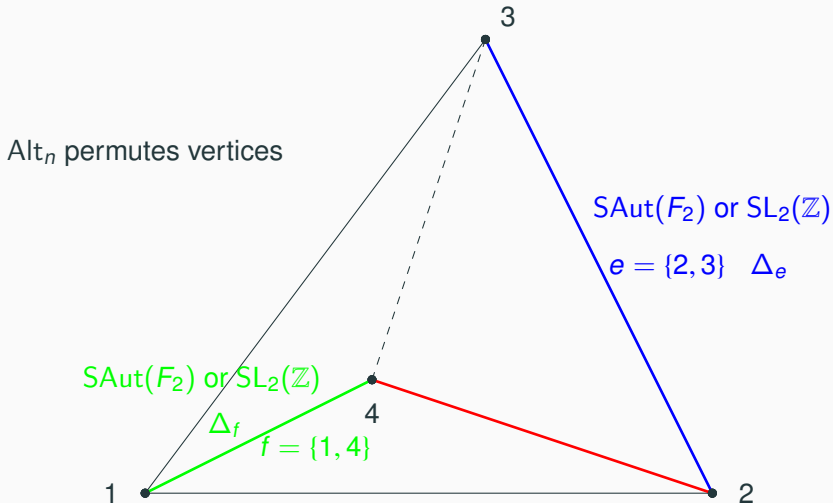


$\text{SAut}(F_2)$ or $\text{SL}_2(\mathbb{Z})$

$e = \{2, 3\} \quad \Delta_e$

Alt_n permutes vertices





In other words, a copy of G_2 is attached at each edge.

Given 2 edges they can

1. coincide
2. be adjacent (share a vertex)
3. be opposite (share no vertices) - corresponding copies of G_2 commute

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$\Delta_n \in \mathbb{R}G_n$ - Laplacian of G_n

For an edge $e = \{i, j\}$ let $S_e = \{s \in S_n : I_n(s) = e\}$ and

$$\Delta_e = |S_e| - \sum_{t \in S_e} t$$

is the Laplacian of the copy of G_2 attached to e

We have $\sigma(\Delta_e) = \Delta_{\sigma(e)}$ for any $\sigma \in \text{Alt}_n$

Δ_e are building blocks of the Laplacians in G_n :

Lemma

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Lemma

$$\Delta_n = \sum_{e \in E_n} \Delta_e = \frac{1}{(n-2)!} \sum_{\sigma \in \text{Alt}_n} \sigma(\Delta_e).$$

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Lemma

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Proposition (Stability for Δ_n)

For $m \geq n \geq 3$ we have

$$\sum_{\sigma \in \text{Alt}_m} \sigma(\Delta_n) = \frac{|\text{Alt}_n|}{(n-2)!} \sum_{\sigma \in \text{Alt}_n} \sigma(\Delta_2) = \binom{n}{2} (m-2)! \Delta_m.$$

Main step in proving property (T) for G_n :

a “stable” decomposition of $\Delta_n^2 - \lambda \Delta_n$

$$\Delta^2 = \left(\sum_e \Delta_e \right) \left(\sum_f \Delta_f \right) = \sum_{e,f} \Delta_e \Delta_f$$

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Define three elements of $\mathbb{R}G_n$:

1. $Sq_n = \sum_{e \in E_n} \Delta_e^2$
2. $Adj_n = \sum_{e \in E_n} \sum_{f \in Adj_n(e)} \Delta_e \Delta_f$
3. $Op_n = \sum_{e \in E_n} \sum_{f \in Op_n(e)} \Delta_e \Delta_f$

Then

$$Sq_n + Adj_n + Op_n = \Delta_n^2$$

Lemma

The elements S_{q_n} and O_{p_n} are sums of squares.

Proof.

Obvious for S_{q_n} .

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Proof.

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For Op_n :

$$\Delta_e = \frac{1}{2} \sum_{t \in S_e} (1-t)^*(1-t)$$

e, f opposite edges then the generators associated to them commute and $\Delta_e \Delta_f$ can be rewritten as a sum of squares using

$$(1-t)^*(1-t)(1-s)^*(1-s) = \left((1-t)(1-s) \right)^* \left((1-t)(1-s) \right)$$

□

Stability for Adj and Op:

Proposition

For $m \geq n \geq 3$ we have

$$\sum_{\sigma \in \text{Alt}_m} \sigma(\text{Adj}_n) = \left(\frac{1}{2} n(n-1)(n-2)(m-3)! \right) \text{Adj}_m$$

Stability for Adj and Op:

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Proposition

For $m \geq n \geq 4$ we have

$$\sum_{\sigma \in \text{Alt}_m} \sigma(\text{Op}_n) = \left(2 \binom{n}{2} \binom{n-2}{n} (m-4)! \right) \text{Op}_m$$

The following allows to prove (T) for $SL_n(\mathbb{Z})$ for $n \geq 3$ and for $SAut(F_n)$, $n \geq 7$.

Theorem

Let $n \geq 3$ and

$$\text{Adj}_n + k \text{Op}_n - \lambda \Delta_n = \sum \xi_i^* \xi_i$$

where $\text{supp } \xi_i \subset B(e, R)$, for some $k \geq 0$, $\lambda \geq 0$.

Then G_m has property (T) for every $m \geq n$ such that

$$k(n - 3) \leq m - 3.$$

Moreover, the Kazhdan radius is bounded above by R .

Proof: Rewrite $\Delta_m^2 - \lambda\Delta_m$ using stability of Op , Adj and Δ .

Eventually arrive at:

$$\begin{aligned}\Delta_m^2 - \frac{\lambda(m-2)}{n-2}\Delta_m &= \text{Sq}_m + \text{Adj}_m + \text{Op}_m - \frac{\lambda(m-2)}{n-2}\Delta_m \\ &= \text{Sq}_m + \left(1 - \frac{k(n-3)}{m-3}\right)\text{Op}_m + \\ &\quad \frac{2}{n(n-1)(n-2)(m-3)!} \sum_{\sigma \in \text{Alt}_m} \sigma(\text{Adj}_n + k\text{Op}_n - \lambda\Delta_n)\end{aligned}$$

When $1 - \frac{k(n-3)}{m-3} \geq 0$ we obtain the claim. □

Results

Theorem

$\text{SAut}(F_n)$ has (T) for $n \geq 6$, with Kazhdan radius 2 and Kazhdan constant estimate

$$\sqrt{\frac{0.138(n-2)}{6(n^2-n)}} \leq \kappa(\text{SAut}_n).$$

Theorem

$\text{SAut}(F_n)$ has (T) for $n \geq 6$, with Kazhdan radius 2 and Kazhdan constant estimate

$$\sqrt{\frac{0.138(n-2)}{6(n^2-n)}} \leq \kappa(\text{SAut}_n).$$

Proof.

The case $n \geq 7$:

$$\text{Adj}_5 + 2 \text{Op}_5 - 0.138\Delta_5$$

certified positive on the ball of radius 2. □

The case $n = 6$ needs a different computation.

The case $n = 5$ so far can only be proven directly.

Currently we can also certify $\text{Adj}_4 + 100 \text{Op}_4 - 0.1\Delta$ in $\text{SAut}(F_4)$ proving (T) for $n \geq 103$.

Theorem

$SL_n(\mathbb{Z})$ has (T) for $n \geq 3$, with Kazhdan radius 2 and Kazhdan constant estimate

$$\sqrt{\frac{0.157999(n-2)}{n^2-n}} \leq \kappa(SL_n(\mathbb{Z})).$$

Proof.

$$\text{Adj}_3 - 0.157999\Delta_3$$

certified positive on the ball of radius 2. □

This gives a new estimate on the Kazhdan constants of $SL_n(\mathbb{Z})$.

We obtain an even better estimate by certifying positivity of

$$\text{Adj}_5 + 1.5 \text{Op}_5 - 1.5 \Delta_5$$

in $\mathbb{R} \text{SL}_5(\mathbb{Z})$:

$$\sqrt{\frac{0.5(n-2)}{n^2-n}} \leq \kappa(\text{SL}_n(\mathbb{Z})) \quad \text{for } n \geq 6.$$

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Previously known bounds:

$$\text{(Kassabov 2005)} \quad \frac{1}{42\sqrt{n} + 860} \leq \kappa(\text{SL}_n(\mathbb{Z})) \leq \sqrt{\frac{2}{n}} \quad \text{(Žuk 1999)}$$

Asymptotically, the new lower bound is 1/2 of the upper bound:

$$\frac{\text{Žuk's upper bound}}{\text{our lower bound}} = 2\sqrt{\frac{n-1}{n-2}} \rightarrow 2 \quad (n \geq 6)$$

Some applications

Product Replacement Algorithm generates random elements in finite groups

Lubotzky and Pak in 2001 showed that property (T) for $\text{Aut}(F_n)$ explains the fast performance of the Product Replacement Algorithm

\implies now proven

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Question (Lubotzky 1994)

Is there a sequence of finite groups such that their Cayley graphs are expanders or not for different generating sets (uniformly bounded)?

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Gilman 1977: $\text{Aut}(F_n)$ for $n \geq 3$ residually alternating

\implies sequences of alternating groups with $\text{Aut}(F_n)$ generators are expanders

This gives an alternative answer to Lubotzky's questions with explicit generating sets