Higher Kazhdan projections & Baum-Connes conjectures

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The Baum-Connes Conjecture

Let G be a countable discrete group. The Baum-Connes conjecture claims that the homomorphism (assembly map)

$$\mu_*\colon K_*^G(\underline{E}G) \to K_*(\mathrm{C}^*_{\mathrm{red}}G) \qquad * = 0, 1$$

is an isomorphism.

- bijectivity $\Rightarrow$ Operator Algebras (K-theory computations)
- surjectivity $\Rightarrow$ Analysis (Kadison-Kaplanski conjecture)
- injectivity $\Rightarrow$ Topology (Novikov conjecture)

A lot of groups satisfy the conjecture, including :

- a-T-menable groups e.g. amenable groups, free groups, $SL(2, \mathbb{Z})$
- hyperbolic groups, hence some property (T) groups e.g. lattices in $Sp(1, n)$

Injectivity is known even for a larger class of groups e.g. linear groups.

Yet un-solved: Does BC hold for $SL(3, \mathbb{Z})$?

Kazhdan's property (T)

A group G has property (T) if whenever a unitary representation of G has almost invariant vectors, it has a non-trivial invariant vector.

Definition
A group G has property (T) iff there exists a projection $p \in \mathrm{C}^*_{\max} G$ such that its image under any unitary representation $(\pi, \mathcal{H})$ of G is the projection $\pi(p)\colon \mathcal{H} \to \mathcal{H}^{\pi(G)}$.

This projection is called Kazhdan projection.

$$\begin{array}{ccc} K_0^G(\underline{E}G) & \xrightarrow{\mu_r} & K_0(\mathrm{C}^*_{\mathrm{red}} G) \\ & \searrow^{\mu_m} & \uparrow \lambda \\ & & K_0(\mathrm{C}^*_{\max} G) \end{array}$$

$\rightsquigarrow$ counterexamples to various versions of the Baum-Connes conjecture

Generalisation of Kazhdan projection

Why a generalisation?

- It is the main source to produce counterexamples to various versions of BC, so it is interesting to study its higher generalisation.
- These projections could live non-trivially in any completion of the amplified group ring.
- Their K-theory class can provide information about the K-theory of the group C^*-algebra.

Laplace operator

Let $G = \langle S \rangle$ be a finitely generated property (T) group. Fix a unitary representation $(\pi, \mathcal{H})$.

Recall: Kazhdan projection provides $p \colon \mathcal{H} \to \mathcal{H}^{\pi(G)}$.

Consider the Markov operator

$$M = \frac{1}{|S|} \sum_{s \in G} s \in \mathbb{C}G$$

$$\text{invariants of } \pi(G) \sim \text{invariants of } \pi(M)$$

Now consider the Laplace operator $\Delta = 1 - M \in \mathbb{C}G$.

$$\ker \pi(\Delta) = \mathcal{H}^{\pi(G)}.$$

Therefore we have

$$\ker \pi(\Delta) = \mathcal{H}^{\pi(G)} = H^0(G, \mathcal{H})$$

Higher Laplace operator

Fix $(\pi, \mathcal{H})$ for G.

$$\ker \pi(\Delta_0) = H^{\pi(G)} = H^0(G, \mathcal{H})$$

Take the free resolution associated to BG for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$

$$\cdots \to \mathbb{Z}G^{\oplus k_2} \to \mathbb{Z}G^{\oplus k_1} \to \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Apply $Hom_{\mathbb{Z}G}(\cdot, \mathcal{H}) \rightsquigarrow$ cochain complex with $Hom(BG^{(n)}, \mathcal{H}) = \mathcal{H}^{\oplus k_n}$

$$\mathcal{H} \xrightarrow{d^0} \mathcal{H}^{\oplus k_1} \to \cdots \to \mathcal{H}^{\oplus k_{n-1}} \xrightarrow{d^{n-1}} \mathcal{H}^{\oplus k_n} \xrightarrow{d^n} \mathcal{H}^{\oplus k_{n+1}} \to \cdots$$

$$\pi(\Delta_0) = d^0 d^{0^*} \curvearrowright \mathcal{H} \qquad d^0 = \begin{pmatrix} 1 - s_1 \\ \vdots \\ 1 - s_m \end{pmatrix}$$

$$\pi(\Delta_n) = d^{n^*} d^n + d^{n-1} d^{n-1^*} \curvearrowright \mathcal{H}^{\oplus k_n}$$

Hodge-de Rham isomorphism

$$\tilde{H}^n(G, \mathcal{H}) \cong \ker \pi(\Delta_n).$$

Higher Kazhdan projection

Fix a unitary representation $(\pi, \mathcal{H})$ for G.

Definition
The higher Kazhdan projection in degree n is the projection $p_n \colon \mathcal{H}^{\oplus k_n} \to \ker \pi(\Delta_n)$.

Remark
This spectral projection $p_n = \lim_{t\to\infty} e^{-t\pi(\Delta_n)}$ always lives in the matrices in the von Neumann algebra generated by $\pi(G)$.

The spectral gap assumption makes the entries lie in the C^*-algebra generated by $\pi(G)$.

Dimensions and Betti numbers

vector subspaces $V \subset \mathbb{C}^n \iff$ orthogonal projection $p \in \mathcal{M}_n(\mathbb{C})$

$$\dim_{\mathbb{C}} V = Tr(p)$$

group von Neumann algebra of G $\quad LG = \overline{\mathbb{C}G} \subset \mathcal{B}(\ell^2 G)$

canonical trace $\tau \colon LG \to \mathbb{C}$ defined by $\tau(\sum_{\text{finite}} c_g g) = c_e$.

- n-th ℓ^2-Betti number $\beta^n_{(2)}(G) = \dim_{LG} \tilde{H}^n(G, \ell^2 G) \in [0, \infty]$.

$$\tilde{H}^n(G, \ell^2 G) \cong p_n(\ell^2 G^{\oplus k_n}) \qquad \text{right LG-module}$$

$$\beta^n_{(2)}(G) = \dim_{LG} \tilde{H}^n(G, \ell^2 G) = \dim_{LG} p_n(\ell^2 G^{\oplus k_n}) = (Tr \otimes \tau)(p_n)$$

Applications to the Baum-Connes conjecture

Proposition
Assume $\lambda(\Delta_n)$ has spectral gap so that the spectral projection $p_n \in M_{k_n}(\mathrm{C}^*_{red}G)$. Then we have that

$$\tau_*([p_n]) = \beta^n_{(2)}(G).$$

In particular if $\beta^n_{(2)}(G) \neq 0$, then $[p_n] \neq 0$ in $K_0(\mathrm{C}^*_{red}G)$.

W.Lück conjectured that under the assumption of surjectivity of the Baum-Connes assembly map for G we have

$$\tau_*(K_0(\mathrm{C}^*_{\mathrm{red}}G)) \subseteq \langle \frac{1}{|F|} \mid F \leq^{finite} G \rangle \subseteq \mathbb{Q}$$

Applications to the Baum-Connes conjecture

Proposition
Assume $\lambda(\Delta_n)$ *has spectral gap so that the spectral projection* $p_n \in M_{k_n}(\mathrm{C}^*_{red}G)$. *If the Baum-Connes assembly map is surjective, then* $\beta^n_{(2)}(G)$ *is rational. In particular if* G *is torsion-free, then it is an integer.*

$\rightsquigarrow$ strategy to find counterexamples to the conjecture

Examples

Lemma (Bader-Nowak)
Let $(\pi, \mathcal{H})$ be unitary representation of G. The operator $\pi(\Delta_n)$ has spectral gap iff $H^n(G, \mathcal{H})$ and $H^{n+1}(G, \mathcal{H})$ are reduced.

- G infinite group. Clearly $\ell^2(G)^\lambda = \{0\}$. $\rightsquigarrow p_0 = 0$
- $G = F_n$. $\lambda(\Delta_1)$ has spectral gap $\rightsquigarrow [p_1] = (n-1)[1]$
- $G = SL(2, \mathbb{Z})$. $\lambda(\Delta_1)$ has spectral gap $\rightsquigarrow \tau_*([p_1]) = \frac{1}{12}$

The coarse Baum-Connes conjecture

X discrete metric space with bounded geometry
$\mathcal{H}$ separable, infinite dimensional Hilbert space
$\mathbb{C}X \subseteq \mathcal{B}(\ell^2(X,\mathcal{H}))$: $*$-algebra of finite propagation operators with compact entries $T_{(x,y)}$.

$$\text{Roe algebra of } X: \quad \mathrm{C}^*X = \overline{\mathbb{C}X} \subseteq \mathcal{B}(\ell^2(X,\mathcal{H}))$$

The coarse Baum-Connes conjecture: (1993 J.Roe) for all X with bounded geometry the assembly map is an isomorphism

$$\mu_*\colon KX_*(X) \to K_*(\mathrm{C}^*X) \qquad * = 0,1.$$

$\rightsquigarrow$ The coarse Baum-Connes conjecture does *have* counterexamples, but also *lots of* confirmed cases.

Applications to the coarse Baum-Connes conjecture

Recall:

- $\beta^n(G) = \dim_{\mathbb{C}} H^n(G, \mathbb{C}) \in \mathbb{N}$.
- $\beta^n_{(2)}(G) = \dim_{LG} \tilde{H}^n(G, \ell^2 G) \in [0, \infty]$

Theorem
Let G be a residually finite and exact group. Let $N = \{N_i\}_I$ be a filtration of finite index normal subgroups of G. Assume that Δ_n has a spectral gap in $\mathcal{M}_k(\mathrm{C}^*_N G)$. If the coarse Baum-Connes assembly map for the box space $Y = \coprod G/N_i$ of G is surjective, then

$$\beta^n_{(2)}(N_i) = \beta^n(N_i)$$

for all but finitely many i.

$\rightsquigarrow$ new strategy to find counterexamples to the conjecture

On the proof

$$\begin{array}{ccccc} & & K_0\left(\dfrac{\prod_i \mathrm{C}^*(G)^{N_i}}{\bigoplus_i \mathrm{C}^*(G)^{N_i}}\right) & & \\ & \nearrow^{\tilde{\mu}_c} & & \searrow^{T^*} & \\ KX_0(Y) & & \uparrow \varphi_* & & \dfrac{\prod \mathbb{R}}{\bigoplus \mathbb{R}} \\ & \searrow_{\mu_c} & & \nearrow_{d_*} & \\ & & K_0(C^*(Y)) & & \end{array}$$

Thanks for listening!