On classifying Hecke algebras associated with right-angled Coxeter groups via K-theory

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virtual Oslo, 30th of March 2021

Plan of the talk

We will introduce (Hecke deformations of) group operator algebras of right angled Coxeter groups, compute their K-theory and discuss the resulting (non-)classification results.

Classifying C^*-algebras by their K-theory

As well-known in the classification of C^*-algebras in the last 50 years the key role has been played by the K-theory, and in particular by the Elliott invariant:

$$Ell(A) = (K_0(A), K_1(A), [1]_{K_0(A)}, K_0(A)^+, T(A), \rho_A).$$

Elliott's programme has seen enormous success in the abstract classification of nuclear C^*-algebras. But K-theory has also been useful in distinguishing algebras beyond the nuclear world.

Theorem (Pimsner-Voiculescu, 1982)

Let $n \in \mathbb{N}$. Then

$$K_0(\mathrm{C}^*_r(\mathbb{F}_n)) = \mathbb{Z}, \quad K_1(\mathrm{C}^*_r(\mathbb{F}_n)) = \mathbb{Z}^n.$$

In particular $\mathrm{C}^*_r(\mathbb{F}_n) \not\approx \mathrm{C}^*_r(\mathbb{F}_m)$ for $n \neq m$.

K-theory for C^*-algebras of free groups

The key role in the result of Pimsner and Voiculescu is played by the following exact sequence:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{0} & K_0(\mathrm{C}_r^*(\mathbb{F}_{n-1})) & \longrightarrow & K_0(\mathrm{C}_r^*(\mathbb{F}_n)) \\ \uparrow & & & & \downarrow \\ K_1(\mathrm{C}_r^*(\mathbb{F}_n)) & \longleftarrow & K_1(\mathrm{C}_r^*(\mathbb{F}_{n-1})) & \longleftarrow & 0 \end{array}$$

Soon later a simpler proof was obtained by Cuntz, who showed also the following.

Theorem (Cuntz, 1982)

The free groups are K-amenable, that is the quotient map $j : \mathrm{C}^*(\mathbb{F}_n) \to \mathrm{C}_r^*(\mathbb{F}_n)$ is a KK-equivalence.

Right-angled Coxeter groups

Coxeter system (W, S): a group W generated by a (finite) set of reflections S, with a function $m : S \times S \mapsto \mathbb{N} \cup \{\infty\}$ which determines the relations:

$$(st)^{m_{s,t}} = e, \quad s, t \in S$$

(we have $m_{s,s} = 1$, $s \in S$).

W is right-angled if $m_{s,t} \in \{2, \infty\}$

In the right-angled case define the commutation graph Γ_W with vertices S and edges

$$E\Gamma_W := \{(s, t) \in S \times S : m_{s,t} = 2\}$$

W is irreducible if the complement of Γ_W is connected; equivalently, W does not decompose as a direct product of two Coxeter groups.

Basic examples; graphs of groups

The simplest case is given by an empty graph: $W = \mathbb{Z}_2^{\star k}$ (note that $k = 1, 2$ lead to amenable groups).

The graph with three vertices and one edge yields $W = (\mathbb{Z}_2 \times \mathbb{Z}_2) \star \mathbb{Z}_2$.

We could keep the finite graph as the way of encoding freeness/commutation, and replace the 'vertex groups' $\mathbb{Z}_2$ by arbitrary groups: this leads to the graph of groups construction. When 'vertex groups' are $\mathbb{Z}$, we obtain right-angled Artin groups.

Right-angled (and not only!) Coxeter groups have strong combinatorial properties, and are related to buildings.

Group ring and its deformation

(W, S) – Coxeter system of a right-angled Coxeter group.

$$\mathbb{C}[W] = \langle \{ T_s, s \in S : \begin{aligned} T_s T_t &= T_t T_s \text{ if } m_{s,t} = 2 \\ T_s &= T_s^* \\ T_s^2 &= 1 \} \rangle \end{aligned}$$

Let $\mathbf{q} = (q_s)_{s \in S} \in (0,1]^S$.

$$\mathbb{C}[W] = \langle \{ T_s, s \in S : \begin{aligned} T_s T_t &= T_t T_s \text{ if } m_{s,t} = 2 \\ T_s &= T_s^* \\ (T_s - 1)(T_s + 1) &= 0 \} \rangle \end{aligned}$$

Group ring and its deformation

(W, S) – Coxeter system of a right-angled Coxeter group.

$$\begin{aligned}\mathbb{C}[W] = \langle\{T_s, s\in S : & T_sT_t = T_tT_s \text{ if } m_{s,t}=2\\ & T_s = T_s^*\\ & (T_s-1)(T_s+1)=0\}\rangle\end{aligned}$$

Let $\mathbf{q} = (q_s)_{s\in S} \in (0,1]^S$.

$$\begin{aligned}\mathbb{C}_{\mathbf{q}}[W] = \langle\{T_s^{\mathbf{q}}, s\in S : & T_s^{\mathbf{q}}T_t^{\mathbf{q}} = T_t^{\mathbf{q}}T_s^{\mathbf{q}} \text{ if } m_{s,t}=2\\ & T_s^{\mathbf{q}} = T_s^{\mathbf{q}*}\\ & (T_s^{\mathbf{q}}-q_s^{\frac{1}{2}})(T_s^{\mathbf{q}}+q_s^{-\frac{1}{2}})=0\}\rangle\end{aligned}$$

$\mathbb{C}_{\mathbf{q}}[W]$ – the $\mathbf{q}$-Hecke algebra of W.

Representing the deformed group ring on $\ell^2(W)$...

$\mathbb{C}_{\mathbf{q}}[W]$ acts on $\ell^2(W)$, as noted by Dymara: for $s \in S$, $w \in W$ put $p_s = \frac{q_s - 1}{\sqrt{q_s}}$ and

$$\pi_{\mathbf{q}}(T_s^{\mathbf{q}})(\delta_w) = \begin{cases} \delta_{sw} & \text{if } |sw| > |w| \\ \delta_{sw} + p_s \delta_w & \text{if } |sw| < |w| \end{cases}$$

For example

$$\begin{aligned} \pi_{\mathbf{q}}(T_s^{\mathbf{q}})\pi_{\mathbf{q}}(T_s^{\mathbf{q}})(\delta_e) &= \pi_{\mathbf{q}}(T_s^{\mathbf{q}})\delta_s = \delta_e + p_s\delta_s = (1 + p_s\pi_{\mathbf{q}}(T_s^{\mathbf{q}}))\delta_e \\ &= (1 + (q_s^{\frac{1}{2}} - q_s^{-\frac{1}{2}})\pi_{\mathbf{q}}(T_s^{\mathbf{q}}))\delta_e \end{aligned}$$

… and operator algebras

Definition

Let (W, S) be a right-angled Coxeter system, $\mathbf{q} = (q_s)_{s \in S} \in (0,1]^S$. The **q**-Hecke von Neumann algebra of W is defined as

$$N_{\mathbf{q}}(W) = \pi_{\mathbf{q}}(\mathbb{C}_{\mathbf{q}}[W])'' \subset B(\ell^2(W)).$$

Similarly the (reduced) **q**-Hecke C^*-algebra of W, $\mathrm{C}^*_{\mathbf{q}}(W)$ is given by the norm closure of $\pi_{\mathbf{q}}(\mathbb{C}_{\mathbf{q}}[W])$ in $B(\ell^2(W))$.

In particular $N_{\mathbf{1}}(W) = L(W)$, $\mathrm{C}^*_{\mathbf{1}}(W) = \mathrm{C}^*_r(W)$.

Algebras $\mathbb{C}[W]$ and $\mathbb{C}_{\mathbf{q}}[W]$ are isomorphic. Thus we can compose π_q with the isomorphism to obtain a new representation of W on $\ell^2(W)$ and get the reduction maps

$$j_{\mathbf{q}} : \mathrm{C}^*(W) \to \mathrm{C}^*_{\mathbf{q}}(W).$$

Hecke von Neumann algebras – factoriality

Theorem (Dymara ('06), Garncarek ('16), Raum+AS ('20))

Assume that W is irreducible, $|S| \geq 3$. The vector state $x \mapsto \langle \delta_e, x\delta_e \rangle$ is a faithful normal trace on $N_{\mathbf{q}}(W)$. If we define $\mathbf{q} : W \to (0,1]$ as the 'multiplicative' (with respect to reduced forms) extension of $\mathbf{q} : S \to (0,1]$ and assume that the $\mathbf{q}$-growth series $\sum_{w \in W} \mathbf{q}_w$ converges, then the projection onto the vector $\sum_{w \in W} (\mathbf{q}_w)^{\frac{1}{2}} \delta_w$ belongs to the centre of $N_{\mathbf{q}}(W)$. If the $\mathbf{q}$-growth series diverges, $N_{\mathbf{q}}(W)$ is a factor.

Results of Dykema show that if $W = (\mathbb{Z}_2)^{\star k}$ (with $k \geq 3$), then $N_{\mathbf{q}}(W)$ are interpolated free group factors.

Graph product structure

The construction of graph products of groups was extended to operator algebras by Caspers and Fima in 2017. It has universal and reduced versions (in the second case the additional data is given by traces of 'vertex' algebras). The graph product has an iterated decomposition in terms of the amalgamated free products.

We have the following identifications:

$$\mathrm{C}^*(W) \cong \prod_{\Gamma_W} C^*(\mathbb{Z}_2),$$

$$\mathrm{C}^*_{\mathbf{q}}(W) \cong \prod_{\Gamma_W} (\mathrm{C}^*(\mathbb{Z}_2), \varphi_s),$$

where φ_s is a weighted measure on two point set with weights $\frac{1}{1+q_s}$, $1 - \frac{1}{1+q_s}$.

K-theory of amalgamated free products (after Fima+Germain)

Thus we need to understand the behaviour of K-theory related to the amalgamated free products.

Theorem (Fima + Germain, 2015)

Let A_1, A_2, B be separable unital C*-algebras, with $B \subset A_1$ and $B \subset A_2$. Then we have an exact sequence

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(A_1) \oplus K_0(A_2) & \longrightarrow & K_0(A_1 *_B A_2) \\ \uparrow & & & & \downarrow \\ K_1(A_1 *_B A_2) & \longleftarrow & K_1(A_1) \oplus K_1(A_2) & \longleftarrow & K_1(B) \end{array}$$

Moreover if the algebras above are equipped with faithful traces, the reduction map $j : A_1 *_B A_2 \to (A_1, \tau_1) *_{(B,\tau)} (A_2, \tau_2)$ is a KK-equivalence.

General K-theory of graph products

The next result is proved via a (slightly delicate) induction based on the theorem of Fima and Germain.

Theorem

Let $\Gamma = (V, E)$ be a finite simplicial graph and let $(A_v, \varphi_v)_{v\in V}$ be a family of unital, separable and (K)-nuclear C^**-algebras with faithful tracial states. Then the reduction map $\prod_\Gamma A_v \to \prod_\Gamma (A_v, \varphi_v)$ is a KK-equivalence.*

This tells us that K-groups themselves do not distinguish the Hecke deformations.

Corollary

Let (W, S) be a right-angled Coxeter system and $\mathbf{q} \in (0,1]^S$. Then $j_{\mathbf{q}} : \mathrm{C}^*(W) \to \mathrm{C}^*_{\mathbf{q}}(W)$ is a KK-equivalence. Each algebra $\mathrm{C}^*_{\mathbf{q}}(W)$ satisfies the UCT.

The main K-theory result

Theorem

Let (W, S) be a right-angled Coxeter system. Denote by Γ_W the associated commutation graph and by $\mathcal{C}$ the set of all cliques (complete subgraphs) of Γ_W (including the empty clique). Then

- *the map $C \mapsto [p_C]$ induces an isomorphism $\mathbb{Z}^{|\mathcal{C}|} \cong K_0(\mathrm{C}^*(W))$;*
- $K_1(\mathrm{C}^*(W)) = 0$.

In particular, $K_0(\mathrm{C}^*(W))$ is a free abelian group generated by the projections from the copies $\mathrm{C}^*(\mathbb{Z}_2) \subset \mathrm{C}^*(W)$ associated with the Coxeter generators $s \in S$.

Elliott invariant revisited

Recall that

$$Ell(A) = (K_0(A), K_1(A), [1]_{K_0(A)}, K_0(A)^+, T(A), \rho_A).$$

For our examples we have so far only discussed $K_1(A)$ and $K_0(A)$; the class of 1 is easy to determine from the last theorem.

Theorem (Fendler, de la Harpe, de Cornulier...)

Assume that W is irreducible, $|S| \geq 3$. Then $C_r^*(W)$ admits unique trace.

So we need to understand the pairing of K_0 with the standard trace.

Trace pairing and non-classification examples

Proposition

Denote by $\mathcal{C}$ the set of cliques of the commutation graph Γ_W. The trace pairing of $K_0(\mathrm{C}^*_{\mathbf{q}}(W))$ with the natural trace is given by

$$\tau_*([p_C]) = \prod_{s\in C} \frac{1}{1+q_s}, \quad c \in \mathcal{C}.$$

Corollary

Consider

$$W_1 = (\mathbb{Z}_2 * \mathbb{Z}_2 \oplus \mathbb{Z}_2) * \mathbb{Z}_2, \quad W_2 = (\mathbb{Z}_2)^{\oplus 2} * (\mathbb{Z}_2)^{\oplus 2}.$$

Then $W_1 \not\simeq W_2$, and

$$Ell(\mathrm{C}^*_r(W_1)) \simeq Ell(\mathrm{C}^*_r(W_2)).$$

Case study – Dykema's free products

Let $n \in \mathbb{N}, n \geq 3$, and assume $q \in (0,1]$, $\mathbf{q} = (q, \ldots, q) \in (0,1]^n$. Put $A_{q,n} = \mathrm{C}^*_{\mathbf{q}}((\mathbb{Z}_2)^{\star n})$.

Theorem (Dykema, 1999)

The Hecke C*-algebra $A_{q,n}$ is

- simple with a unique trace if and only if $q > \frac{1}{n-1}$,
- simple with exactly two extremal traces if $q = \frac{1}{n-1}$, and
- a direct sum of a one dimensional algebra with a simple C*-algebra having a unique trace if $q < \frac{1}{n-1}$.

The simple algebras appearing above have stable rank one.

Case study – Dykema's free products

Consider the unordered Elliott invariant of a C^*-algebra A:

$$Ell_{unord}(A) = (K_0(A), K_1(A), [1]_{K_0(A)}, T(A), \rho_A).$$

Recall the notation $A_{q,n} = \mathrm{C}^*_{\mathbf{q}}((\mathbb{Z}_2)^{\star n})$.

Theorem

Let $q_1, q_2 \in (\frac{1}{n-1}, 1]$. *Then*

$$Ell_{unord}(A_{q_1,n}) \simeq Ell_{unord}(A_{q_2,n})$$

if and only if

- *either* $q_1 = q_2$, *or*
- *both* q_1, q_2 *are rational and* $\frac{1}{1+q_1}, \frac{1}{1+q_2}$ *have the same order in* $\mathbb{Q}/\mathbb{Z}$.

When $q_1, q_2 \in (0, \frac{1}{n-1}]$ the unordered Elliott invariants of $A_{q_1,n}$ and $A_{q_2,n}$ are isomorphic.

And what about the order?

Before, when we were considering say $C^*_r((\mathbb{Z}_2 * \mathbb{Z}_2 \oplus \mathbb{Z}_2) * \mathbb{Z}_2)$, we talked about the full Elliott invariants. The last theorem only talks about the unordered versions...

Empirical observations

When one can classify a class of C^*-algebras via Elliott invariant, the order in K_0 is determined by traces. When we can identify explicitly the order in K_0, it is determined by traces.

Dykema and Rørdam showed in 1998 that for free products satisfying Avitzour conditions the order is determined by traces. Beyond that we do not know what happens!

What else do we know about q-Hecke operator algebras?

Let (W, S) be a right-angled irreducible Coxeter system with $|S| \geq 3$,
$\mathbf{q} = (q_s)_{s \in S} \in (0, 1]^S$.

- the C^*-algebra $\mathrm{C}^*_{\mathbf{q}}(W)$ is non-nuclear, exact (Caspers+Klisse+Larsen);
- when $\mathbf{q}$ is 'close to 1', the C^*-algebra $\mathrm{C}^*_{\mathbf{q}}(W)$ is simple, has unique trace (Caspers+Klisse+Larsen);
- when the $\mathbf{q}$-growth series diverges, the factors $N_{\mathbf{q}}(W)$ are non-injective, have weak-* completely contractive approximation property and the Haagerup property; sometimes have no Cartan subalgebras (Caspers).

And what we would like to know

- is $C^*_{\mathbf{q}}(W)$ is simple and does it have a unique trace in the whole range of divergence of the **q**-growth series? See Mario Klisse's talk later today!
- when the **q**-growth series converges, $C^*_{\mathbf{q}}(W)$ admits a character χ; is then the kernel of χ a simple C^*-algebra?
- are the algebras $C^*_{\mathbf{q}}((\mathbb{Z}_2)^{\star k})$ pairwise non-isomorphic in the cases where their (unordered) Elliott invariants coincide? Or rather, how to prove it?

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