Quantum differentials on cross product Hopf algebras

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Quantum Riemannian geometry by quantum groups approach:

- Differentials on an algebra $A$ is $A - A$-bimodule $\Omega^1$ (space of 1-forms):
  - $d : A \rightarrow \Omega^1$ (differential map) s.t. $d(ab) = (da)b + a(db)$ (Leibniz rule)
  - $\Omega^1 = \text{span}\{ad\}b\}$ (surjectivity)
  - $\ker d = k.1$ (connectedness, conditional).

- Exterior algebra means a DGA $\Omega = \bigoplus_{n \geq 0} \Omega^n$ on $A$ generated by $\Omega^0 = A, dA$ with
  - $d : \Omega^n \rightarrow \Omega^{n+1}$ s.t. $d(\omega\tau) = (d\omega)\tau + (-1)^{|\omega|}\omega d\tau$ (graded-Leibniz rule)
  - $d^2 = 0$. 
\( \Omega^1 \) is left (resp. right) covariant if it is a left (resp. right) \( A \)-comodule algebra with \( \Delta_L : \Omega \to A \otimes \Omega^1, \Delta_L d = (\text{id} \otimes d)\Delta \) (resp. \( \Delta_R : \Omega^1 \otimes \Omega^1 \otimes A, \Delta_R d = (d \otimes \text{id})\Delta \)).

\( \Omega^1 \) is bicovariant if it is both left and right covariant.

Can be extended to have \( \Omega \) left/right/bicovariant.

[Brzeziński '93] \( \Omega^1 \) bicovariant \( \Rightarrow \) \( \Omega \) super-Hopf algebra (\( \mathbb{Z}_2 \)-graded)

\[
\Delta_*|_{\Omega_0} = \Delta, \quad \Delta_*|_{\Omega^1} = \Delta_L + \Delta_R
\]

\[
\Delta_*(dadb) = \Delta_*(da)\Delta_*(db)
\]
Motivation and Problem

- Knowing only $\Omega^1$ and $\Omega^2$, we can build elements of noncommutative geometry (metric, connection, torsion, curvature) algebraically on the DGA.
- In nice cases, we can recover the Dirac operators as in Connes’ approach but does not require it as axiom.
- Fundamental problem: there will be many $\Omega^1$ and $\Omega^2$ on a given Hopf algebra $A$.
- Woronowicz construction of bicovariant $\Omega^1$:

$$\Omega^1 \cong A \otimes A^+ / I; \quad A^+ = \ker \epsilon; \quad I : \text{ad-stable right ideal}$$

- No general result known, but for some cases $\Omega^1$ are classified:
  - coquasitriangular Hopf algebra $A$ (Bauman, Schmidt ’98)
  - the Sweedler-Taft algebra $U_q(b_+)$ (Oeckl ’99).
We introduce a method (different from Woronowicz) to construct DGAs on all main type of cross (co)product Hopf algebras:

- On double cross product $A \leftrightarrow A \triangleleft H \leftrightarrow H$.
- On double cross coproduct $A \leftrightarrow A \triangleright H \rightarrow H$.
- On bicrossproduct $A \leftrightarrow A \blacktriangleleft H \rightarrow H$.
- On biproduct $A \leftrightarrow A \blacktriangleright B$ (Here $B$ is a braided Hopf algebra)
Assumption: $\Omega(A), \Omega(H), \Omega(B)$ are strongly bicovariant exterior algebras.

Their differentials are built by using their super version, e.g. $\Omega(A \triangleright H) := \Omega(A) \triangleright \Omega(H)$ gives a strongly bicovariant exterior algebra on $A \triangleright H$, etc.

We do not classify all $\Omega^1$ but the resulting exterior algebra is natural in the sense it (co)acts on its factor differentiably.

In this talk, we will focus on differentials on biproduct $A \ll B$. 
Braided Hopf algebras

Def (Majid '90s) : Let $C$ be braided monoidal category. $B \in C$ is a braided Hopf algebra if it is algebra + coalgebra + antipode $S : B \to B$ s.t.

\[ \Delta(bc) = b_{(1)} \Psi(b_{(2)} \otimes c_{(1)})c_{(2)}. \]
If $A$ is ordinary Hopf algebra and $B$ is braided Hopf algebra in $\mathcal{M}_A^A$ crossed module (or Drinfeld-Radford-Yetter module) category, then there is a biproduct $A \ltimes B$ (or the Radford-Majid bosonisation of $B$) built in $A \otimes B$ with

\[(a \otimes b)(c \otimes d) = ac^{(1)} \otimes (b \triangleright c^{(2)})d\]

\[\Delta(a \otimes b) = a^{(1)} \otimes b^{(1)}(0) \otimes a^{(2)} b^{(1)}(1) \otimes b^{(2)}\]

for all $a, c \in A$, $b, d \in B$.

Example: $\mathbb{C}_q[P] = \mathbb{C}_q[GL_2] \ltimes \mathbb{C}_q^2 \cong \mathbb{C}_q[SL_3]/(t^i_j | i > j)$ a deformation of maximal parabolic $P \subset SL_3$
Super Crossed Modules

- Let $A$ be a super Hopf algebra, i.e. $A = A_0 \oplus A_1$.
- Let $V = V_0 \oplus V_1$ be a super right $A$-crossed module over a super-Hopf algebra $A$ if
  1. $V$ is a super right $A$-module by $\triangleleft : V \otimes A \to V$
  2. $V$ is a super right $A$-comodule by $\Delta_R : V \to V \otimes A$ denoted $\Delta_R v = v^{(0)} \otimes v^{(1)}$, such that

$$\Delta_R (v \triangleleft a) = (-1)^{|v^{(1)}||a^{(1)}|+|v^{(1)}||a^{(2)}|+|a^{(1)}||a^{(2)}|} v^{(0)} \triangleleft a^{(2)} \otimes (Sa^{(1)}) v^{(1)} a^{(3)}$$

for all $v \in V$ and $a \in A$.
- The category $\mathcal{M}_A^A$ of super right $A$-crossed modules is a prebraided category with the braiding $\Psi : V \otimes W \to W \otimes V$,

$$\Psi (v \otimes w) = (-1)^{|v||w^{(0)}|} w^{(0)} \otimes (v \triangleleft w^{(1)})$$

and braided if $A$ has invertible antipode.
Strongly bicovariant exterior algebras

(Majid - Tao '15) \( \Omega \) is **strongly bicovariant** if it is:

- a super-Hopf algebra with super-degree given by the grade mod 2
- super-coproduct \( \Delta_* \) grade preserving and restricting to the coproduct of \( A \)
- \( d \) is a **super coderivation** in the sense

\[
\Delta_* d \omega = (d \otimes \text{id} + (-1)^{|\text{id}|} \text{id} \otimes d) \Delta_* \omega
\]

**Lemma (Majid - Tao '15)**

\( \Omega \) **Strongly bicovariant** \( \Rightarrow \) \( \Omega \) **bicovariant**

**Lemma**

\( \Omega(A), \Omega(H) \) **strongly bicovariants** \( \Rightarrow \) \( \Omega(A \otimes H) := \Omega(A) \otimes \Omega(H) \) is **strongly bicovariant on** \( A \otimes H \) with \( d = d_A \otimes \text{id} + (-1)^{|\text{id}|} \text{id} \otimes d_H \).
Differentiable Coaction

- Let $A$ be Hopf algebra, $\Omega(A)$ be its exterior algebra.
- Let $B \in M^A$ be comodule algebra, $\Omega(B)$ is $A$-covariant, i.e. the coaction $\Delta_R : \Omega(B) \to \Omega(B) \otimes A$ (denoted by $\Delta_R \eta = \eta^{(0)} \otimes \eta^{(1)}$) is a comodule map.
- $\Delta_R$ is differentiable if it extends to a degree-preserving map $\Delta_{R^*} : \Omega(B) \to \Omega(B) \otimes \Omega(A)$ of exterior algebras such that

$$d_B \Delta_{R^*} = d \Delta_{R^*}$$

or explicitly

$$\Delta_{R^*} d_B \eta = d_B \eta^{(0)^*} \otimes \eta^{(1)^*} + (-1)^{|\eta|} \eta^{(0)^*} \otimes d_A \eta^{(1)^*},$$

where $\Delta_{R^*} \eta = \eta^{(0)^*} \otimes \eta^{(1)^*} \in \Omega(B) \otimes \Omega(A)$. 


Let $A$ be Hopf algebra, $\Omega(A)$ be its exterior algebra.

Let $B \in \mathcal{M}_A$ be a module algebra, $\Omega(B)$ is $A$-covariant, i.e. the action $\triangleleft : \Omega(B) \otimes A \to \Omega(B)$ is a module map.

The action $\triangleleft$ is **differentiable** if it extends to a degree preserving map $\triangleleft : \Omega(B) \otimes \Omega(A) \to \Omega(A)$ such that

$$d_B \triangleleft = \triangleleft d$$

or explicitly

$$d_B(\eta \triangleleft \omega) = (d_B \eta) \triangleleft \omega + (-1)^{|\eta|} \eta \triangleleft (d_A \omega)$$

for all $\eta \in \Omega(B), \omega \in \Omega(A)$. 
Super Biproducts

Assumption:

1. $B$ is a braided Hopf algebra in $\mathcal{M}_A$ s.t. they form $A \ltimes B$.
2. $\Omega(B) \in \mathcal{M}_A$ with differentiable action and coaction.
3. $\Omega(B)$ is a super braided Hopf algebra in super crossed module category $\mathcal{M}_{\Omega(A)}^{\Omega(A)}$ with $d_B$ a super coderivation.

Then we have super biproduct $\Omega(A) \ltimes \Omega(B)$

$$(\omega \otimes \eta)(\tau \otimes \xi) = (-1)^{|\eta||\tau(1)}\omega_{\tau(1)} \otimes (\eta \lhd \tau(2))\xi$$

$$\Delta_*(\omega \otimes \eta) = (-1)^{|\omega(2)||\eta(1)^{(0)}|} \omega_{(1)} \otimes \eta_{(1)^{(0)}}^* \otimes \omega_{(2)} \eta_{(1)^{(1)}}^* \otimes \eta_{(2)}$$

for all $\omega, \tau \in \Omega(A)$ and $\eta, \xi \in \Omega(B)$. 
Differentials by Super Biproducts

**Theorem**

1. Under the assumptions above, $\Omega(A \bowtie B) := \Omega(A) \bowtie \Omega(B)$ is a strongly bicovariant exterior algebra on $A \bowtie B$ with differential map

   $$d(\omega \otimes \eta) = d_A \omega \otimes \eta + (-1)^{\lvert \omega \rvert} \omega \otimes d_B \eta$$

   for all $\omega \in \Omega(A)$, $\eta \in \Omega(B)$.

2. The canonical $\Delta_R : B \to B \otimes A \bowtie B$ given by

   $$\Delta_R b = b^{(1)}_0 \otimes b^{(1)}_1 \otimes b^{(2)}_0$$

   is differentiable, i.e it extends to

   $\Delta_R^* : \Omega(B) \to \Omega(B) \otimes \Omega(A \bowtie B)$ by

   $$\Delta_R^* \eta = \eta^{(0)*}_1 \otimes \eta^{(1)*}_1 \otimes \eta^{(2)}_2$$
Let \( R \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \) be \( q \)-Hecke (\( PR \) has two eigen-values).

Let \( A(R) \) be an FRT algebra generated by \( t = (t^i_j) \) with

\[
Rt_1t_2 = t_2t_1R, \quad \Delta t = t \otimes t
\]

\( A = A(R)[D^{-1}] \), \( D \in A(R) \) central, grouplike.

\( \Omega(A(R)) \) has

\[
(dt_1)t_2 = R_{21}t_2dt_1R, \quad dt_1dt_2 = -R_{21}dt_2dt_1R
\]

\[
dD^{-1} = -D^{-1}(dD)D^{-1}, \quad \Delta_* dt = dt \otimes t + t \otimes dt
\]

Let \( V(R) \in \mathcal{M}^A \) a braided covector algebra generated by \( x = (x_i) \) with \( qx_1x_2 = x_2x_1R \), \( \Delta_R x = x \otimes t \)

\( \Omega(V(R)) \in \mathcal{M}^{\Omega(A)} \) has

\[
(dx_1)x_2 = x_2dx_1qR, \quad -dx_1dx_2 = dx_2dx_1qR, \quad \Delta_R^* dx = dx \otimes t + x \otimes dt
\]
Theorem

Let $A = A(R)[D^{-1}]$ with $R$ q-Hecke and $V(R)$ the right-covariant braided covector algebra. Then $\Omega(V(R))$ is a super-braided-Hopf algebra with $x_i, dx_i$ primitive in $M^{\Omega(A)}_{\Omega(A)}$ with $\Delta_R^* dx = dx \otimes t + x \otimes dt$ and

\[
x_1 \triangledown t_2 = x_1 q^{-1} R_{21}^{-1}, \quad dx_1 \triangledown t_2 = dx_1 q^{-1} R
\]

\[
x_1 \triangledown dt_2 = (q^{-2} - 1) dx_1 P, \quad dx_1 \triangledown dt_2 = 0,
\]

and $\Omega(A \triangledown V(R)) := \Omega(A) \triangledown \Omega(V(R))$ with

\[
x_1 t_2 = t_2 x_1 q^{-1} R_{21}^{-1}, \quad dx_1 . t_2 = t_2 dx_1 q^{-1} R,
\]

\[
x_1 dt_2 = dt_2 . x_1 q^{-1} R_{21}^{-1} + (q^{-2} - 1) t_2 dx_1 P, \quad dx_1 dt_2 = -dt_2 dx_1 q^{-1} R
\]

\[
\Delta x = 1 \otimes x + x \otimes t, \quad \Delta^*_dx = 1 \otimes dx + dx \otimes t + x \otimes dt.
\]
Differential on Quantum Parabolic Group

- For \( R = R_{gl_2} \), then \( A = \mathbb{C}_q[GL_2] \) generated by
  \( t^1_{11} = a, t^1_{12} = b, t^2_{1} = c, t^2_{2} = d \) with

  \[
  ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd
  \]

  \[
  da - ad = (q - q^{-1})bc, \quad ad - q^{-1}bc = da - qcb = D
  \]

  \[
  \Delta t^i_{j} = t^i_{k} \otimes t^k_{j}
  \]

- Let \( V(R) = \mathbb{C}^2_q \in \mathcal{M}_{\mathbb{C}_q[GL_2]} \) a two-dimensional quantum plane
  with \( x_2x_1 = q, \Delta x_i = 1 \otimes x_i + x_i \otimes 1 \) and \( \Delta_R x_i = x_j \otimes t^j_i \)
Differential on Quantum Parabolic Group

- $\Omega(\mathbb{C}_q^2)$ has

\[(dx_i)x_i = q^2 x_i dx_i, \quad (dx_1)x_2 = qx_2 dx_1\]

\[(dx_2)x_1 = qx_1 dx_2 + (q^2 - 1)x_2 dx_1\]

\[(dx_i)^2 = 0, \quad dx_2 dx_1 = -q^{-1} dx_1 dx_2\]

- By requiring differentiability on $\Delta_R : \mathbb{C}_q^2 \to \mathbb{C}_q^2 \otimes \mathbb{C}_q[GL_2]$, it enforces us to use the following $\Omega(\mathbb{C}_q[GL_2])$

\[da.a = q^2 ada, \quad da.b = qbd a, \quad db.a = qadb + (q^2 - 1)bda\]

\[dd.a = add, \quad db.c = cdb + (q - q^{-1})dd d, \quad \text{etc.}\]

$\Delta_* dt^i_j = dt^i_k \otimes t^k_j + t^i_k \otimes dt^k_j$
Differential on Quantum Parabolic Group

\(\Omega(\mathbb{C}^2_q)\) is a super braided Hopf algebra in \(\mathcal{M}_{\Omega(\mathbb{C}_q[GL_2])}\) by

\[
x_1 \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-2} x_1 & (q^{-2} - 1)x_2 \\ 0 & q^{-1} x_1 \end{pmatrix}, \quad x_2 \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \ldots
\]

\[
x_1 \triangleleft \begin{pmatrix} da & db \\ dc & dd \end{pmatrix} = \begin{pmatrix} (q^{-2} - 1)dx_1 & (q^{-2} - 1)dx_2 \\ 0 & 0 \end{pmatrix}
\]

\[
dx_1 \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} dx_1 & 0 \\ 0 & q^{-1} dx_1 \end{pmatrix}
\]

\[
x_2 \triangleleft \begin{pmatrix} da & db \\ dc & dd \end{pmatrix} = \ldots, \quad dx_2 \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \ldots
\]

\[dx_i \triangleleft dt^k = 0, \quad \Delta_R x_i = x_j \otimes t^j_i, \quad \Delta_{R^*} dx_i = dx_j \otimes t^j_i + x_j \otimes dt^j_i\]

\[\Delta x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \Delta_{*} dx_i = dx_i \otimes 1 + 1 \otimes dx_i.\]
Then (i) \( \Omega(\mathbb{C}_q[P]) = \Omega(\mathbb{C}_q[GL_2] \ltimes \mathbb{C}^2) := \Omega(\mathbb{C}_q[GL_2]) \ltimes \Omega(\mathbb{C}_q^2) \) with sub-exterior algebras \( \Omega(\mathbb{C}_q[GL_2]) \), \( \Omega(\mathbb{C}_q^2) \) and cross relations and super coproduct

\[
\begin{align*}
\Delta x_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} q^{-2}a & q^{-1}b + (q^{-2} - 1)a \\ q^{-2}c & q^{-1}d + (q^{-2} - 1)c \end{pmatrix}, \quad \Delta x_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \cdots \\
\Delta \star (dx_1) &= 1 \otimes dx_1 + \Delta_R(dx_i), \quad \Delta \star (dx_2) = 1 \otimes dx_2 + \Delta_R(dx_i).
\end{align*}
\]

(ii) \( \Delta_R : \mathbb{C}_q[GL_2] \to \mathbb{C}_q[GL_2] \otimes \mathbb{C}_q[P] \) is differentiable

\[
\begin{align*}
\Delta_R x_i &= 1 \otimes x_i + x_j \otimes t^i_j, \quad \Delta_R^* dx_i = 1 \otimes dx_i + dx_j \otimes t^i_j + x_j \otimes dt^i_j.
\end{align*}
\]
The canonical coactions $\Delta_R : A \to A \otimes H \triangleright \triangleleft A$ and $\Delta_L : H \to H \triangleright \triangleleft A \otimes H$ are differentiable, i.e. they extend to

$$\Delta_{R}^{*} : \Omega(A) \to \Omega(A) \otimes \Omega(H) \triangleright \triangleleft \Omega(A)$$

$$\Delta_{L}^{*} : \Omega(H) \to \Omega(H) \triangleright \triangleleft \Omega(A) \otimes \Omega(H)$$

making $\Omega(H)$ and $\Omega(A)$ super $\Omega(H \triangleright \triangleleft A)$-comodule algebras.

The canonical coaction $\Delta_R : H \to H \otimes A \triangleright \triangleleft H$ is differentiable, i.e. it extends to

$$\Delta_{R}^{*} : \Omega(H) \to \Omega(H) \otimes \Omega(A) \triangleright \triangleleft \Omega(H)$$

making $\Omega(H)$ a super $\Omega(A \triangleright \triangleleft H)$-comodule algebra.
Overview

- $A \otimes H$ acts on f.d. $A^*$ as module algebra by

$$\left(\phi \triangleleft h\right)(a) = \phi(h \triangleright a), \quad \phi \triangleleft a = \langle \phi(1), a \rangle \phi(2),$$

Similarly for a left action on $H^*$. However, for differentiability, we would need $\Omega(A^*)$ or $\Omega(H^*)$ to be specified.
Thank you for your attention