On the uniform convergence of Cesaro averages for $C^*$-dynamical systems

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Let \((\mathcal{A}, \Phi)\) be a \(C^*-\)dynamical system based on an identity-preserving \(*\)-endomorphism \(\Phi\) of the unital \(C^*\)-algebra \(\mathcal{A}\). We study the uniform convergence of Cesaro averages

\[ M_{a,\lambda}(n) := \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} \Phi^k(a), \quad a \in \mathcal{A}, \]

uniformly for values \(\lambda\) in the unit circle.

For such a purpose, we define a spectral set \(\sigma_{\text{pp}}^{(\text{ph},f)}(\Phi) \subset \mathbb{T}\) canonically associated to the given dynamical system, and show that

\[ \lim_{n \to +\infty} M_{a,\lambda}(n) = 0 \]

whenever \(\lambda \in \mathbb{T} \setminus \sigma_{\text{pp}}^{(\text{ph},f)}(\Phi)\).

If in addition, if \((\mathcal{A}, \Phi)\) is uniquely ergodic w.r.t. the fixed-point algebra, then we can provide conditions for which the sequence \((M_{a,\lambda}(n))_n\)
uniformly converges, even for $\lambda \in \sigma_{pp}^{ph}(\Phi)$, providing the formula of such a limit.

To end, we also discuss some simple examples arising from quantum probability, the first one not enjoying the property to be uniquely ergodic w.r.t. the fixed point subalgebra, and the second one satisfying such a strong ergodic property, to which our results apply. Other more involved examples coming from noncommutative geometry (i.e. the noncommutative 2-torus) can be exhibited.

The present talk is based on the papers:

(i) F. Fidaleo *Uniform Convergence of Cesaro Averages for Uniquely Ergodic C*-Dynamical Systems*, Entropy 20 (2018), 987.
On the uniform convergence of ergodic averages

With $\mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ we denote the unit circle of the complex plane. It is homeomorphic to the interval $[0, 2\pi)$ by $\theta \in [0, 2\pi) \mapsto e^{-i\theta}$, after identifying the endpoints 0 and $2\pi$.

A (discrete) $C^*$-dynamical system is a triplet $(\mathcal{A}, \Phi, \varphi)$ consisting of a $C^*$-algebra, a positive map $\Phi : \mathcal{A} \to \mathcal{A}$ and a state $\varphi \in \mathcal{S}(\mathcal{A})$ such that $\varphi \circ \Phi = \varphi$. Consider the Gelfand-Naimark-Segal
(GNS for short) representation \((\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)\). If in addition
\[
\varphi(\Phi(a)^*\Phi(a)) \leq \varphi(a^*a), \quad a \in \mathcal{A},
\]
then there exists a unique contraction \(V_{\varphi,\Phi} \in \mathcal{B}(\mathcal{H}_\varphi)\) such that \(V_{\varphi,\Phi}\xi_\varphi = \xi_\varphi\) and
\[
V_{\varphi,\Phi}\pi_\varphi(a)\xi_\varphi = \pi_\varphi(\Phi(a))\xi_\varphi, \quad a \in \mathcal{A}.
\]
The quadruple \((\mathcal{H}_\varphi, \pi_\varphi, V_{\varphi,\Phi}, \xi_\varphi)\) is called the covariant GNS representation associated to the triplet \((\mathcal{A}, \Phi, \varphi)\).

If \(\Phi\) is multiplicative, hence a \(*\)-homomorphism, then \(V_{\varphi,\Phi}\) is an isometry with final range \(V_{\varphi,\Phi}V_{\varphi,\Phi}^*\), the orthogonal projection onto the subspace \(\pi_\varphi(\mathcal{A})\xi_\varphi\).

Now we specialise the matter to \(C^*\)-dynamical systems \((\mathcal{A}, \Phi, \varphi)\) such that \(\mathcal{A}\) is a unital \(C^*\)-algebra with unity \(1 = I_{\mathcal{A}}\), and \(\Phi\) is multiplicative and unital preserving.
Denote by $\mathcal{A}_\Phi := \{ a \in \mathcal{A} \mid \Phi(a) = a \}$ the fixed-point subalgebra, and

$$\sigma_{pp}(\Phi) := \{ \lambda \in \mathbb{T} \mid \lambda \text{ is an eigenvalue of } \Phi \}$$

the set of the peripheral eigenvalues of $\Phi$ (i.e. the peripheral pure-point spectrum), with $\mathcal{A}_\lambda$ the relative eigenspaces. Obviously, $1 \in \mathcal{A}_\Phi = \mathcal{A}_1$.

Analogously, for the invariant state $\varphi \in S(\mathcal{A})$, consider the pure-point peripheral spectrum

$$\sigma_{pp}(V_{\varphi,\Phi}) := \{ \lambda \in \mathbb{T} \mid \lambda \text{ is an eigenvalue of } V_{\varphi,\Phi} \}$$

of $V_{\varphi,\Phi}$. Denote with $P_\lambda \in \mathcal{B}(\mathcal{H}_\varphi)$ the orthogonal projection onto the eigenspace generated by the eigenvectors associated to $\lambda \in \mathbb{T}$, with the convention $P_\lambda = 0$ if $\lambda \notin \sigma_{pp}(V_{\varphi,\Phi})$.

Let $S(\mathcal{A})^\Phi$ be the (convex, $*$-weakly compact) set of all invariant states under the action of
the \(*\)-endomorphism \(\Phi\), and define the full peripheral pure point spectrum as
\[
\sigma_{pp}^{(ph,f)}(\Phi) := \bigcup \{\sigma_{pp}(V_\varphi, \Phi) \mid \varphi \in S(\mathcal{A})^\Phi\}.
\]
Notice that, it is a spectral set canonically associated to the \(C^*\)-dynamical system \((\mathcal{A}, \Phi)\).
We have the following

**Theorem:**
Let \((\mathcal{A}, \Phi)\) be a \(C^*\)-dynamical system. For \(\lambda \in \mathbb{T} \setminus \sigma_{pp}^{(ph,f)}(\Phi)\), we have

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} \Phi^k(a) = 0, \quad (1)
\]
uniformly, for each \(a \in \mathcal{A}\).

**Proof** (sketch): If (1) does not hold, then there exists an invariant state \(\varphi\), for which the spectral measure of \(V_{\varphi, \Phi}\) has an atom corresponding to \(\lambda = e^{-i\theta}\). But this contradicts \(\lambda \notin \sigma_{pp}^{(ph,f)}(\Phi)\).
Uniquely ergodic $C^*$-dynamical systems

The $C^*$-dynamical system $(\mathcal{A}, \Phi)$ is said to be uniquely ergodic w.r.t. the fixed point subalgebra if the ergodic average $\frac{1}{n} \sum_{k=0}^{n-1} \Phi^k(a)$ converges for each fixed $a \in \mathcal{A}$. In such a situation, there is a unique invariant conditional expectation $E_1 : \mathcal{A} \to \mathcal{A}_1$ given by

$$E_1(x) := \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k(a), \quad a \in \mathcal{A}.$$ 

If $\mathcal{A}_1 = \mathbb{C}$, then $E_1(x) = \varphi(x) \mathbb{1}_{\mathcal{A}}$ with $\varphi \in \mathcal{S}(\mathcal{A})$ is an invariant state, which is indeed unique (i.e. $\mathcal{S}(\mathcal{A})\Phi$ is the singleton $\{\varphi\}$). Therefore, the $C^*$-dynamical system $(\mathcal{A}, \Phi)$ is said to be uniquely ergodic if there exists only one invariant state $\varphi$ for the dynamics induced by $\Phi$. For a uniquely ergodic $C^*$-dynamical system, we simply write $(\mathcal{A}, \Phi, \varphi)$ by pointing out that $\varphi \in \mathcal{S}(\mathcal{A})$ is the unique invariant state.
Here, there are some standard results relative to uniquely ergodic $C^*$-dynamical systems.

**Proposition:**
Let the $C^*$-dynamical system $(\mathcal{A}, \Phi, \varphi)$ be uniquely ergodic. Then $\sigma_{\text{pp}}^\text{ph}(\Phi)$ is a subgroup of $\mathbb{T}$, and the corresponding eigenspaces $\mathcal{A}_\lambda$, $\lambda \in \sigma_{\text{pp}}^\text{ph}(\Phi)$ are generated by a single unitary $u_\lambda$.

We have the following immediate corollary of the above result:

**Corollary:**
Let the $C^*$-dynamical system $(\mathcal{A}, \Phi, \varphi)$ be uniquely ergodic. Then $\sigma_{\text{pp}}^\text{ph}(\Phi) \subset \sigma_{\text{pp}}^\text{ph}(V_{\varphi, \Phi})$.

The main result involving the uniquely ergodic $C^*$-dynamical systems is the following
Theorem:
Let $\mathcal{A}, \Phi, \varphi$ be a uniquely ergodic $C^*$-dynamical system. Fix $\lambda \in \sigma_{pp}^{ph}(\Phi) \cup \sigma_{pp}^{ph}(V_{\varphi}, \Phi)^c$. Then for each $a \in \mathcal{A}$,

$$
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k(a) \lambda^{-k} = \varphi(u^*_\lambda a) u_\lambda,
$$

uniformly for $n \to +\infty$, where $u_\lambda \in \mathcal{A}_\lambda$ is any unitary eigenvalue corresponding to $\lambda \in \sigma_{pp}^{ph}(\Phi)$ (with the convention that if $\lambda \in \sigma_{pp}^{ph}(V_{\varphi}, \Phi)^c$, then $u_\lambda = 0$).

Proof:
First consider the case $\lambda \in \sigma_{pp}^{ph}(\Phi)$, and take a unitary eigenvector $u_\lambda \in \mathcal{A}_\lambda$, unique up to a phase-factor. Since $\Phi$ is multiplicative, we have

$$
\varphi(u^*_\lambda a) u_\lambda = u_\lambda \lim_n \left( \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k(u^*_\lambda a) \right)
= \lim_n \left( \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k(a) \lambda^{-k} \right).
$$
The case $\lambda \notin \sigma_{pp}^{ph}(V_\varphi,\Phi)$ follows by the result in the previous section because
\[
\sigma_{pp}^{(ph, f)}(\Phi) = \sigma_{pp}^{ph}(V_\varphi,\Phi).
\]
\[\square\]

We can exhibit simple examples based on the tensor product construction, for which $\sigma_{pp}^{ph}(\Phi) \subsetneq \sigma_{pp}^{ph}(V_\varphi,\Phi)$ and for some $a \in \mathcal{A}$ and $\lambda \in \sigma_{pp}^{ph}(V_\varphi,\Phi) \setminus \sigma_{pp}^{ph}(\Phi)$ such that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} \Phi^k(a)
\]
fails to exist, even in the weak topology. More complicated examples of this phenomenon are constructed by using the cross-product construction (i.e. a "genuine" noncommutative framework) coming from the noncommutative 2-torus.

Concerning the $C^*$-dynamical systems $(\mathcal{A}, \Phi)$ which are uniquely ergodic w.r.t. the fixed
point subalgebra $\mathcal{A}_1 \supseteq \mathbb{C} \mathbf{1}_\mathcal{A}$, for $\lambda \in \sigma_{\text{pp}}^{\text{ph}}(\Phi)$ we can provide conditions on $\mathcal{A}_\lambda$ for which there exists a norm one projection $E_\lambda : \mathcal{A} \to \mathcal{A}_\lambda$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} \Phi^k(a) = E_\lambda(a), \quad a \in \mathcal{A}. \quad (2)$$

More precisely, suppose that $u \in \mathcal{A}_\lambda$ is an isometry or a co-isometry. We can prove that

$$\{ \lambda \in \sigma_{\text{pp}}^{\text{ph}}(\Phi) \mid \mathcal{A}_\lambda \text{ contains an isometry or a co-isometry} \} \subset \sigma_{\text{pp}}^{(\text{ph},f)}(\Phi).$$

In addition,

(i) $\mathcal{A} \ni x \mapsto E_\lambda(x) := E_1(xu^*)u \in \mathcal{A}_\lambda$ (isometry case),

(ii) $\mathcal{A} \ni x \mapsto E_\lambda(x) := uE_1(u^*x) \in \mathcal{A}_\lambda$ (co-isometry-case),
and (2) holds true.

Notice that, for uniquely ergodic $C^*$-dynamical systems $(\mathcal{A}, \Phi, \varphi)$ (i.e. when $\mathcal{A}_1 = \mathbb{C} \mathbb{1}_\mathcal{A}$), this is always the case because

$$E_\lambda(a) = \varphi(u_\lambda^* x) u_\lambda = \varphi(x u_\lambda^*) u_\lambda,$$

where $u_\lambda$ is the unique unitary (up to a phase-factor) generating $\mathcal{A}_\lambda$.

**Examples**

We are listing simple examples coming from quantum probability for which the obtained results apply. More complicated examples can be obtained by considering skew-products on the noncommutative 2-torus.
the monotone case

We consider the $C^*$-dynamical system $(m, s)$ where $m$ is the concrete $C^*$-algebra generated by the identity $I = 1_m$ and the monotone creators $\{m_n^\dagger \mid n \in \mathbb{Z}\}$ acting on the monotone Fock space $\Gamma_{mon}(\ell^2(\mathbb{Z}))$ on $\ell^2(\mathbb{Z})$. It has the structure $m = a + CI$ where $I \notin a$, and thus the state at infinity $\omega_\infty$ is meaningful. The one-step shift $s$ is defined on generators as $s(m_j^\dagger) = m_{j+1}^\dagger$, $j \in \mathbb{Z}$.

The main properties of $(m, s)$ are summarised as follows:

- for the fixed-point subalgebra, $m^s = CI$,

- the set of all invariant states

$$S(m)^s = \{(1 - t)\omega_0 + t\omega_\infty \mid t \in [0, 1]\}$$
is the convex combination of the vacuum state $\omega_0$ and the state at infinity $\omega_{\infty}$.

Therefore, $(m, s)$ cannot be uniquely ergodic w.r.t. the fixed-point subalgebra. Indeed, it can be viewed by direct inspection because

$$
\frac{1}{n} \sum_{k=0}^{n-1} s^k(m_l m_l^\dagger) = \frac{1}{n} \sum_{k=0}^{n-1} m_{l+k} m_{l+k}^\dagger \downarrow P_{e_0},
$$

the self-adjoint projection onto the subspace generated by the vacuum vector $e_0$. But such a convergence in the strong operator topology, cannot be uniform.

We can check that

$$
\sigma_{pp}(\alpha) = \{1\} = \sigma^{(f)}_{pp}(\alpha), *
$$

and thus for each $x \in m$ and $\lambda \in \mathbb{T} \setminus \{1\}$,

$$
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} s^k(x) = 0,
$$

*For $*$-automorphisms, all spectra are included in the unit circle $\mathbb{T}$, and therefore we omit the suffix "ph".
uniformly.

the boolean case

We consider the $C^*$-dynamical system $(\mathfrak{b}, s)$, where $\mathfrak{b}$ is the concrete $C^*$-algebra generated by the identity and the boolean creators $\{b_n^\dagger \mid n \in \mathbb{Z}\}$ acting on the boolean Fock space $\Gamma_{\text{boole}}(\ell^2(\mathbb{Z}))$ on $\ell^2(\mathbb{Z})$, and (with an abuse of notation) $s$ is the one-step shift acting on generators as $s(b_j^\dagger) = b_{j+1}^\dagger$, $j \in \mathbb{Z}$.

It was shown that $\mathfrak{b}$ is nothing but the $C^*$-algebra $\mathcal{K}(\ell^2(\{\emptyset\} \sqcup \mathbb{Z})) + \mathbb{C} I$ generated by all compact operators acting on $\Gamma_{\text{boole}}(\ell^2(\mathbb{Z})) = \ell^2(\{\emptyset\} \sqcup \mathbb{Z})$ and the identity $I := 1_{\ell^2(\{\emptyset\} \sqcup \mathbb{Z})}$. The shift is therefore generated by the adjoint action $\text{ad}_V$, with $V$ defined on the canonical basis $\{e_{\emptyset}\} \sqcup \{e_j \mid j \in \mathbb{Z}\}$ of $\ell^2(\{\emptyset\} \sqcup \mathbb{Z})$ by

$$V e_{\emptyset} = e_{\emptyset}, \quad V e_j = e_{j+1}, \quad j \in \mathbb{Z}.$$
Furthermore, we have:

- for the fixed point subalgebra, $b_1 \equiv b^s = CP_{e_0} \oplus CP_{e_0}^\perp$;

- the set of all invariant states

$$S(b)^s = \{(1-t)\omega_0 + t\omega_\infty \mid t \in [0,1]\}$$

is the convex combination of the vacuum state $\omega_0$ and the state at infinity $\omega_\infty$.

- with $a \in \mathcal{K}(\ell^2(\{0\} \cup \mathbb{Z})$,

$$b \ni A + bI \mapsto \mathcal{E}_1(a + bI) := (\langle Ae_0, e_0 \rangle + b) P_{e_0} + b P_{e_0}^\perp \in b_1$$

is a conditional expectation, invariant under the shift $s$;

- the $C^*$-dynamical system $(b, s)$ is uniquely mixing (and therefore uniquely ergodic) w.r.t.
the fixed-point subalgebra with conditional expectation \( \mathcal{E}_1 \).

Notice that the set \( \mathcal{S}(b)^s \) of the boolean invariant states has the same structure as that \( \mathcal{S}(m)^s \) of the monotone invariant ones. Furthermore, \( \sigma_{pp}(s) = \{1\} = \sigma_{pp}^{(f)}(s) \) as for the monotone case. Differently to \( (m,s) \), the \( C^* \)-dynamical system \( (b,s) \) is uniquely ergodic w.r.t. the fixed-point subalgebra. Therefore, for the convergence of ergodic averages we have for \( x \in b \) and \( \lambda \in \mathbb{T} \),

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} s^k(x) = \begin{cases} 
\mathcal{E}_1(x) & \text{if } \lambda = 1, \\
0 & \text{if } \lambda \neq 1.
\end{cases}
\]

In order to provide an example for which the involved spectra are non trivial, we consider the tensor product construction of the previous boolean \( C^* \)-dynamical system with the irrational rotations on the unit circle.

For the irrational number \( \theta \in (0, 1) \), denote by \( R_\theta \) the rotation on \( \mathbb{T} \) of the angle \( 2\pi \theta \). Let
Let $(\mathcal{A}, \alpha)$ be the tensor product $C^*$-dynamical system, where $\mathcal{A} = C(\mathbb{T}) \otimes \mathbb{b} = C(\mathbb{T}; \mathbb{b})$,

\[ \alpha(f)(z) := s(f(e^{2\pi i\theta}z)), \quad z \in \mathbb{T}, \; f \in C(\mathbb{T}; \mathbb{b}), \]

Finally, for each $f \in C(\mathbb{T}; \mathbb{b})$ define

\[ E_1(f) := \left( \int \otimes \mathcal{E}_1 \right)(f) = \int \mathcal{E}_1(f(z)) \frac{dz}{2\pi i z}. \]

Notice that, with $1 \in C(\mathbb{T})$ the constant function identically equal to 1, $E_1$ is projecting onto the fixed-point subalgebra $\mathcal{A}_1 = 1 \otimes \mathbb{b} \sim \mathbb{b}$.

The main results involving such a dynamical system, whose spectral sets are non-trivial by construction, are:

**Proposition:**

The $C^*$-dynamical system $(\mathcal{A}, \alpha)$ is uniquely ergodic w.r.t. the fixed-point subalgebra with expectation $E_1$. 
In addition,
\[ \sigma_{pp}(\alpha) = \{e^{2\pi i l \theta} \mid l \in \mathbb{Z}\} = \sigma_{pp}^{(f)}(\alpha), \]
where, for \( \lambda_l = e^{2\pi i l \theta} \in \sigma_{pp}(\alpha) \), \( \mathcal{A}_{\lambda_l} = u_l \mathcal{A}_1 = \mathcal{A}_1 u_l \), with \( u_l(z) = z^l \otimes I \in \mathcal{A}_{\lambda_l} \) unitary.

Finally, for \( f \in \mathcal{A} \) and \( \lambda \in \mathbb{T} \),
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n-1} \lambda^{-k} \alpha_k^k(f) = \begin{cases} 
\left( \mathcal{E}_1(f(z)) \frac{dz}{2\pi i z^{l+1}} \right) u_l & \text{if } \lambda = \lambda_l, \\
0 & \text{if } \lambda \neq \lambda_l.
\end{cases}
\]