# Interpolating products of quantum groups

**Daniel Gromada**, Universität des Saarlandes, Saarbrücken

Joint with Moritz Weber

arXiv:1907.08462

Supported by SFB-TRR 195

# Compact matrix quantum groups

- A **compact matrix quantum group** is a compact quantum group with a distinguished **fundamental representation**.
- That is, a pair $G = (A, u)$, where $A$ is a C*-algebra and $u \in M_N(A)$ such that

1. the elements $u_{ij}$ $i, j = 1, \ldots, N$ generate $A$,
2. the matrices $u$ and $u^t = (u_{ji})$ are invertible,
3. the map $\Delta\colon A \to A \otimes_{\min} A$ defined as $\Delta(u_{ij}) := \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$ extends to a $*$-homomorphism.

# Products of quantum groups



For classical matrix groups $G, H$, we have

$$G \times H = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \mid g \in G, h \in H \right\}$$

# Products of quantum groups



For classical matrix groups $G, H$, we have

$$G \times H = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \mid g \in G, h \in H \right\}$$

For $G = (A, u)$, $H = (B, v)$ CMQGs:

- The **tensor product** $G \times H = (A \times B, u \oplus v)$ [Wang 1995]
- The **free product** $G * H = (A * B, u \oplus v)$ [Wang 1995]

# Products of quantum groups

For classical matrix groups $G, H$, we have

$$G \times H = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \mid g \in G, h \in H \right\}$$

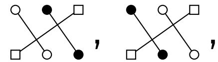
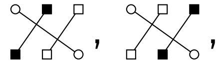
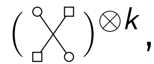
For $G = (A, u)$, $H = (B, v)$ CMQGs:

- The **tensor product** $G \times H = (A \times B, u \oplus v)$ [Wang 1995]
- The **free product** $G * H = (A * B, u \oplus v)$ [Wang 1995]

**Question:** Are there some quantum groups between $G \times H$ and $G * H$?

# Interpolating products

We define quantum subgroups of $G * H$ by imposing new relations:

- $G \mathbin{\times\!\!\!\!\diagdown} H$: $\quad ab^*x = xab^*, \quad a^*bx = xa^*b,$ 
- $G \mathbin{\times\!\!\!\!\diagup} H$: $\quad ax^*y = x^*ya, \quad axy^* = xy^*a,$ 
- $G \times_0 H := G \mathbin{\times\!\!\!\!\diagdown} H \cap G \mathbin{\times\!\!\!\!\diagup} H$ (i.e. by all four together)
- $G \times_{2k} H$: $\quad a_1x_1 \cdots a_kx_k = x_1a_1 \cdots x_ka_k,$ 

where $a, b, a_1, \ldots, a_k \in \{u_{ij}\}$ and $x, y, x_1, \ldots, x_k \in \{v_{ij}\}$.

**Theorem** [G., Weber]**:** For $l$ a divisor of $k$, we have

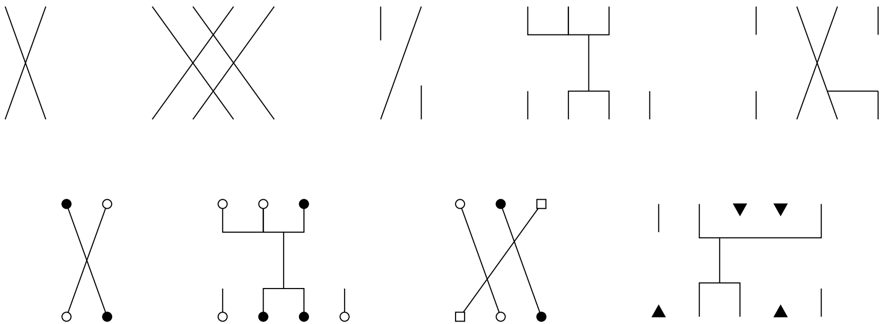
$$G * H \begin{array}{c} \supset \\ \supset \end{array} \begin{array}{c} G \mathbin{\times\!\!\!\!\diagdown} H \\ G \mathbin{\times\!\!\!\!\diagup} H \end{array} \begin{array}{c} \supset \\ \supset \end{array} G \times_0 H \supset G \times_{2k} H \supset G \times_{2l} H \supset G \times_2 H = G \times H,$$

The last three inclusions are strict if and only if the degree of reflection of both $G$ and $H$ is different from one.

# Partitions

- Partitions stand for quantum group relations.
- Categories of partitions define quantum groups.
- Nice combinatorial way how to look for new quantum groups.

**Examples:**



# Partitions with extra singletons

- can be used to describe quantum subgroups $G \subset O_N^+ * \hat{\mathbb{Z}}_2$

$$O_N^+ = (C(O_N^+), v), \qquad \hat{\mathbb{Z}}_2 = (C^*(\mathbb{Z}_2), r)$$

$$\longrightarrow \quad G = (C(G), v \oplus r) \subset O_N^+ * \hat{\mathbb{Z}}_2 = (C(O_N^+) * C^*(\mathbb{Z}_2), v \oplus r)$$

 $\leftrightarrow v,$  $\leftrightarrow r$

 $\leftrightarrow v_{ij} r = r v_{ij} \quad \longrightarrow \quad O_N^+ \times \hat{\mathbb{Z}}_2$

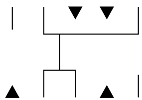
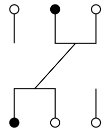
 $\leftrightarrow v_{ij} v_{kl} r = r v_{ij} v_{kl} \quad \longrightarrow \quad O_N^+ \times\!\!\!\times \hat{\mathbb{Z}}_2$

$($$)^{\otimes k} \leftrightarrow v_{i_1 j_1} \cdots v_{i_k j_k} r = r v_{i_1 j_1} \cdots v_{i_k j_k} \quad \longrightarrow \quad O_N^+ \times_{2k} \hat{\mathbb{Z}}_2$

# Classification of partitions with extra singletons

- We have a correspondence

partitions with extra singletons $\leftrightarrow$ two-colored partitions

 $\leftrightarrow$ 

$$G \subset O_N^+ * \hat{\mathbb{Z}}_2 \leftrightarrow \tilde{G} \subset U_N^+$$

$$v \oplus r \leftrightarrow \tilde{v} = vr$$

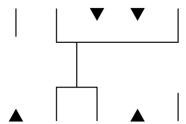
- There are some classification results for two-colored categories
  - Non-crossing [Tarrago–Weber, 2018]
  - Globally colorized [D. G., 2018]
  - Pairs with neutral color sum [Mang–Weber, 2019]
  - Non-hyperoctahedral categories [Mang–Weber, yesterday]
  - Ongoing research by Maaßen, Mang, Weber. . .

# Summary

- We introduced **new products** of compact matrix quantum groups interpolating the free and the tensor product

$$G * H \;\begin{matrix} \supset & G \mathbin{\times\!\!\!\!\diagdown} H & \supset \\ \supset & G \mathbin{\times\!\!\!\!\diagup} H & \supset \end{matrix}\; G \times_0 H \supset G \times_{2k} H \supset G \times_{2l} H \supset G \times_2 H = G \times H,$$

- We adapted the framework of partition categories to describe the quantum subgroups of $O_N^+ * \hat{\mathbb{Z}}_2$ by introducing **partitions with extra singletons**



Thank you for your attention!