Duality theory for C*-crossed products

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Abstract

These are the lecture notes for a mini-course given by the author at the University of Oslo in June 2019. They present the essentials of duality for C*-dynamical systems, and briefly outline recent research done jointly with S. Kaliszewski and Magnus Landstad on the Baum-Connes Conjecture.

1 Introduction

In these lectures I'll present the essentials of C^*-dynamical systems and crossed products. A C^*-dynamical system is an action of a locally compact group on a C^*-algebra. The crossed product of the system is a C^*-algebra with the same representation theory. I'll focus on crossed-product duality, where the goal is to recover the action from the crossed product. We will see how to do this up to tensoring with the compacts. When the group is abelian this can be accomplished using the Pontryagin dual group. But for nonabelian groups we need coactions, which are dual to actions. This introduces a lack of symmetry into the theory, so we will also need to see how to recover a coaction from its crossed product.

I'll emphasize the method of universal properties throughout, which allows us to define the main components in terms of how they work rather than how they can be constructed. I'll also freely use the pedagogical method of "black boxes", meaning I'll develop much of the theory axiomatically rather than from scratch, and in most cases I'll give at most an outline of the proof.

At the end I'll briefly outline an application of the theory to recent efforts to "fix" the Baum-Connes Conjecture.

2 Prelude on multipliers

All ideals of C^*-algebras will be tacitly assumed to be closed and two-sided. By a homomorphism between C^*-algebras we will always mean a *-homomorphism. We use the same convention for representations and isomorphisms. Unless otherwise stated, A and B will denote C^*-algebras and H a (complex) Hilbert space. $\operatorname{Aut} A$ will denote the group of all automorphisms of A, i.e., of all isomorphisms from A into itself.

Definition 2.1. An ideal I of a C^*-algebra A is *essential* if there is no nonzero ideal J of A such that $IJ = 0$.

Definition 2.2. A *multiplier algebra* of a C^*-algebra A is a C^*-algebra $M(A)$ containing A as an essential ideal such that for every C^*-algebra B containing A as an essential ideal there is a unique homomorphism $\pi\colon B \to M(A)$ extending the inclusion map $A \hookrightarrow M(A)$. A *multiplier* of A is an element of $M(A)$.

Every C^*-algebra has a multiplier algebra: represent A faithfully and nondegenerately on a Hilbert space H, then take

$$M(A) = \{T \in B(H) : TA \cup AT \subseteq A\},$$

the *idealizer* of A in $B(H)$. This makes it obvious that every multiplier algebra is unital.

Remark 2.3. The universal property of $M(A)$ makes it unique up to (unique) isomorphism. In practice, we choose one and call it "the" multiplier algebra of A.

Example 2.4. If X is a locally compact Hausdorff space, then $M(C_0(X)) = C_b(X)$.

Example 2.5. If H is a Hilbert space, then $M(\mathcal{K}(H)) = B(H)$, where $\mathcal{K}(H)$ denotes the C^*-algebra of compact operators on H.

Definition 2.6. The *strict topology* on $M(A)$ is the locally convex topology generated by the seminorms

$$m \mapsto \|ma\| \quad \text{and} \quad m \mapsto \|am\| \quad \text{for } a \in A,$$

and is the weakest topology making the maps $m \mapsto ma$ and $m \mapsto am$ continuous.

$M(A)$ is a complete topological vector space with the strict topology, and is in fact the strict completion of A.

Definition 2.7. A homomorphism $\pi\colon A \to M(B)$ is *nondegenerate* if $\pi(A)B = B$.

Strictly (sorry!) speaking, we mean that $\overline{\operatorname{span}}\{\pi(A)B\} = B$, but by the Hewitt-Cohen theorem this implies that actually $\pi(A)B = B$, i.e., every element of B can be factored as $\pi(a)b$ for some $a \in A$ and $b \in B$.

Example 2.8. A representation $\pi\colon A \to B(H)$ on Hilbert space is nondegenerate in the usual sense (namely, $\pi(A)H = H$) if and only if the homomorphism $\pi\colon A \to M(\mathcal{K}(H))$ is nondegenerate in the sense of Definition 2.7.

Every nondegenerate homomorphism $\pi\colon A \to M(B)$ has a unique extension to a homomorphism $\overline{\pi}\colon M(A) \to M(B)$, which is necessarily unital and strictly continuous. In practice we usually just abuse the notation by writing π for $\overline{\pi}$.

3 Locally compact groups

Definition 3.1. A *locally compact group* is a group G equipped with a locally compact Hausdorff topology such that the operations $(s, t) \mapsto st$ and $s \mapsto s^{-1}$ are continuous.

Example 3.2. $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$.

Example 3.3. Matrix groups, Lie groups.

Example 3.4. Non-examples: the unitary group $\mathcal{U}(H)$ of a Hilbert space and the unitary multipliers $\mathcal{U}M(A)$ of a C^*-algebra. These are usually given the strong operator and strict topologies, respectively, and are not locally compact except in the finite-dimensional case.

G will denote a locally compact group from now on.

Definition 3.5. A *representation* $U\colon G \to M(A)$ is a strictly continuous unitary homomorphism. We also say that U is a representation *in* $M(A)$.

Example 3.6. A strong (equivalently, weak) operator continuous representation on Hilbert space

$$U\colon G \to \mathcal{U}(H) \subseteq B(H) = M(\mathcal{K}(H)).$$

Definition 3.7. A *group algebra* of G is a C^*-algebra $C^*(G)$ together with a representation $i_G\colon G \to M(C^*(G))$ such that for every representation $U\colon G \to M(A)$ there is a unique nondegenerate homomorphism $\pi_U\colon C^*(G) \to M(A)$, called the *integrated form* of U, such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{i_G} & M(C^*(G)) \\ & {\scriptstyle U}\searrow & \Big\downarrow {\scriptstyle !\,\pi_U} \\ & & M(A) \end{array}$$

commutes.

In order to prove the existence of the group algebra, we need a construction. We follow a strategy that is ubiquitous in C^*-algebra theory: we first define a *-algebra using properties of G, then complete this in a universal C^*-norm. The *-algebra will be the compactly supported continuous functions $C_c(G)$, equipped with a convolution-type product. For this we need a very special kind of measure on G, called a *Haar measure*, with the following properties: it is a nonzero regular Borel measure μ that is invariant under left translation:

$$\mu(sE) = \mu(E) \quad \text{for all } s \in G \text{ and Borel } E \subseteq G.$$

This invariance can also be characterized by

$$\int_G f(st)\,d\mu(t) = \int_G f(t)\,d\mu(t) \quad \text{for all } f \in C_c(G), s \in G.$$

Happily, every locally compact group has a Haar measure. Moreover, it is unique up to positive multiples. Just like for Lebesgue integration, we write $\int_G f(t)\,dt$ rather than $\int_G f(t)\,d\mu(t)$.

Example 3.8. Lebesgue measure on $\mathbb{R}$, or more generally $\mathbb{R}^n$.

Example 3.9. Counting measure on $\mathbb{Z}$.

Example 3.10. Normalized arclength measure on the circle group $\mathbb{T}$. Here "normalized" means we scale so that the measure of the whole group is 1. This is a common convention for compact groups.

On some groups, a Haar measure is not invariant under right translation. However, $E \mapsto \mu(Es)$ is another Haar measure, so there is $\Delta(s) > 0$ such that

$$\mu(Es) = \Delta(s)\mu(E).$$

This gives a continuous homomorphism $\Delta\colon G \to (0,\infty)$, called the *modular function* of G. The measure $E \mapsto \mu(E^{-1})$ is right-invariant, and in fact we could just as well have taken right-invariance for the definition of Haar measure. But left-invariance is the adopted convention. Groups whose Haar measures are also right-invariant are precisely those whose modular function is trivial in the sense that $\Delta(s) = 1$ for all $s \in G$, and these groups are unsurprisingly called *unimodular*. Abelian, discrete, and compact groups, among others, are all unimodular. But the $ax + b$ group is not.

Theorem 3.11. *$C^*(G)$ exists and is unique up to isomorphism*

Outline of proof. The construction uses the convolution *-algebra $C_c(G)$ with operations

$$\begin{aligned}(f * g)(s) &= \int_G f(t)g(t^{-1}s)\,dt\\ f^*(s) &= \Delta(s)^{-1}\overline{f(s^{-1})}.\end{aligned}$$

Every representation U of G integrates to a (nondegenerate) representation π_U of $C_c(G)$:

$$\pi_U(f) = \int_G f(s)U_s\,ds,$$

and routine estimates show that the operator norm of $\pi_U(f)$ is bounded above by the 1-norm $\|f\|_1$. Thus there is a largest C^*-norm on $C_c(G)$, and we can take $C^*(G)$ to be the completion. Then every representation of G has an *integrated form*, which is a nondegenerate representation of $C^*(G)$.

We get a representation $i_G\colon G \to M(C^*(G))$ by extending continuously the action on $C_c(G)$ given by

$$\begin{aligned}(sf)(t) &= f(s^{-1}t)\\ (fs)(t) &= \Delta(s^{-1})f(ts^{-1}).\end{aligned}$$

Every (nondegenerate) representation of $C^*(G)$ extends canonically to $M(C^*(G))$, and then composing with i_G gives a representation of G. It's not hard to verify that this gives an inverse to the integrated-form construction. Essential uniqueness now follows as an easy exercise — this is a key feature of the method of universal properties. □

In practice, we suppress the notation π_U, and just write U for both the representation of G and its integrated form. We also identify G with its image under i_G.

Definition 3.12. The *regular representation* λ of G on $L^2(G)$ is by left translation:

$$(\lambda_s\xi)(t) = \xi(s^{-1}t) \quad \text{for } s, t \in G, \xi \in L^2(G).$$

The *reduced group algebra* $C_r^*(G)$ is $\lambda(C^*(G))$.

To avoid confusion, frequently we call $C^*(G)$ the *full group algebra.* Actually, there is another version of the regular representation using right translation instead of left:

Definition 3.13. The *right regular representation* ρ of G on $L^2(G)$ is by right translation:

$$(\rho_s\xi)(t) = \xi(ts)\Delta(s)^{1/2}.$$

The delta function Δ appears in the formula for ρ_s to make the operator preserve the 2-norm. To avoid confusion, sometimes we call λ the left regular representation.

I state the following theorem, without any comment about proof, because it is so convenient. $C^*(G)$ is nice because of the universal property, but is obnoxiously large. $C_r^*(G)$ is easier to work with since it is concretely represented on a familiar Hilbert space, so it is good to know that in many cases the two group algebras coincide.

Theorem 3.14. *The integrated form $\lambda\colon C^*(G) \to C_r^*(G)$ is faithful, which we write as "$C^*(G) = C_r^*(G)$", if and only if G is amenable.*

Amenable means there is a positive translation-invariant linear functional on $L^\infty(G)$, and is satisfied when G is compact or abelian, for example. The most famous examples of nonamenable groups are the free groups F_n for $n > 1$.

When G is abelian, the group algebra $C^*(G)$ is commutative, and it follows, essentially by definition, that *characters* of G (that is, continuous homomorphisms from G to the circle group $\mathbb{T}$) correspond bijectively to characters of $C^*(G)$ (that is, nonzero homomorphisms from $C^*(G)$ to $\mathbb{C}$). The characters of G form a group with pointwise multiplication, and with an appropriate topology this turns out to be another locally compact abelian group, called the *Pontryagin dual group* and denoted by $\widehat{G}$. The Gelfand transform from $C^*(G)$ to $C_0(\widehat{G})$ (which by Gelfand's theorem is an isomorphism) is called the *Fourier transform* of G. The Pontryagin duality theorem says that doing it twice gets you back:

$$\widehat{\widehat{G}} = G.$$

Example 3.15. By the elementary theory of classical Fourier series and Fourier transforms, $\widehat{\mathbb{Z}} = \mathbb{T}$ and $\widehat{\mathbb{R}} = \mathbb{R}$, so $\widehat{\mathbb{T}} = \mathbb{Z}$ and $\mathbb{R}$ is self-dual.

4 Actions

Definition 4.1. An *action* (A, α) of G (also called a *C^*-dynamical system*) is a homomorphism $\alpha\colon G \to \operatorname{Aut} A$ such that the maps $s \mapsto \alpha_s(a)\colon G \to A$ are continuous for every $a \in A$.

Example 4.2. Every action on a commutative C^*-algebra arises from an action of G on a locally compact Hausdorff space, by Gelfand's theorem. Two fundamental special cases are $(C_0(G), \mathrm{lt})$ and $(C_0(G), \mathrm{rt})$, where G acts on itself by left or right translation, respectively.

Example 4.3. The *trivial* action of G on $\mathbb{C}$ is self-explanatory, and more generally we can let G act trivially on any C^*-algebra.

Example 4.4. An *inner* (or *unitary*) action $(A, \operatorname{Ad} U)$ of G is determined by a representation $U\colon G \to M(A)$, and is given by

$$\operatorname{Ad} U_s(a) = U_s a U_s^* \quad \text{for } s \in G, a \in A.$$

Example 4.5. A $\mathbb{Z}$-action is determined by a single automorphism of A.

Definition 4.6. A *covariant representation* $(\pi, U)\colon (A, \alpha) \to M(B)$ of an action of G consists of a nondegenerate homomorphism $\pi\colon A \to M(B)$ and a representation $U\colon G \to M(B)$ such that

$$\pi \circ \alpha_s = \operatorname{Ad} U_s \circ \pi \quad \text{for } s \in G.$$

We also say that (π, U) is a covariant representation of (A, α) *in* $M(B)$. If B is the compact operators $\mathcal{K}(H)$ on a Hilbert space H, we say that (π, U) is a covariant representation *on* H.

Example 4.7. For $f \in C_0(G)$ let $M_f \in B(L^2(G))$ be the multiplication operator

$$(M_f \xi)(s) = f(s)\xi(s) \quad \text{for } \xi \in L^2(G), s \in G.$$

Then (M, λ) is a covariant representation of the action $(C_0(G), \mathrm{lt})$ in $M(\mathcal{K}(L^2(G)) = B(L^2(G))$, alternatively on $L^2(G)$.

Definition 4.8. A *crossed product* of (A, α) is a C^*-algebra $A \rtimes_\alpha G$ together with a covariant representation $(i_A, i_G)\colon (A, \alpha) \to M(A \rtimes_\alpha G)$ such that for every covariant representation

$(\pi, U)\colon (A, \alpha) \to M(B)$ there is a unique nondegenerate homomorphism $\pi \times U\colon A \rtimes_\alpha G \to M(B)$, called the *integrated form* of (π, U), such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & M(A \rtimes_\alpha G) & \xleftarrow{i_G} & G \\ & \searrow^{\pi} & \Big\downarrow {\scriptstyle !\,\pi\times U} & \swarrow_{U} & \\ & & M(B) & & \end{array}$$

commutes.

Theorem 4.9. *$A \rtimes_\alpha G$ exists and is unique up to isomorphism.*

Outline of proof. The argument is similar to Theorem 3.11, using a convolution algebra $C_c(G, A)$ with operations

$$\begin{aligned} (f * g)(s) &= \int_G f(t)\alpha_t(g(t^{-1}s))\,dt & \qquad f^*(s) &= \Delta(s^{-1})\alpha_s(f(s^{-1}))^* \\ (i_A(a)f)(s) &= af(s) & (i_G(t)f)(s) &= \alpha_t(f(t^{-1}s)) \\ (fi_A(a))(s) &= f(s)\alpha_s(a) & (fi_G(t))(s) &= f(st^{-1})\Delta(t^{-1}). \end{aligned}$$

The integrated form $\pi \times U$ is given on $C_c(G, A)$ by

$$\pi \times U(f) = \int_G \pi(f(s))U_s\,ds.$$

□

In the display ending the above proof it's a good exercise to derive the bottom row from the one above it using adjoints.

Definition 4.10. Given two actions (A, α) and (B, β), a homomorphism $\pi\colon A \to B$ is $\alpha - \beta$ *equivariant* if

$$\pi \circ \alpha_s = \beta_s \circ \pi \quad \text{for } s \in G.$$

An *isomorphism* $\pi\colon (A, \alpha) \to (B, \beta)$ is an equivariant homomorphism that is also an isomorphism of A onto B. If such an isomorphism exists, we say that the actions $(A, \alpha), (B, \beta)$ are *isomorphic*, written $(A, \alpha) \simeq (B, \beta)$.

Example 4.11. If $\pi\colon A \to B$ is $\alpha - \beta$ equivariant, then

$$(i_B \circ \pi, i_G^\beta)\colon (A, \alpha) \to M(B \rtimes_\beta G)$$

is a covariant homomorphism, and we write the integrated form as

$$\pi \rtimes G\colon A \rtimes_\alpha G \to B \rtimes_\beta G.$$

For $f \in C_c(G, A)$ we have

$$(\pi \rtimes G)(f) = \pi \circ f \in C_c(G, B).$$

Example 4.12. $C^*(G) = \mathbb{C} \rtimes_{\text{trivial action}} G$. This is a special case of the following example.

Example 4.13. $A \rtimes_{\text{trivial action}} G \simeq A \otimes_{\max} C^*(G)$, since in this case covariant representations are just commuting homomorphisms, which by universal properties correspond to homomorphisms of the maximal tensor product.

Definition 4.14. The *regular representation* $\big((\mathrm{id} \otimes M) \circ \widetilde{\alpha}, 1 \otimes \lambda\big)$ is constructed as follows: first define a homomorphism

$$\widetilde{\alpha}\colon A \to M(A \otimes C_0(G))$$

by

$$\widetilde{\alpha}(a)(b \otimes f)(s) = \alpha_{s^{-1}}(a) b f(s).$$

Next, let M be the representation of $C_0(G)$ on $L^2(G)$ by multiplication operators, as in Example 4.7. Now regard both M and λ as representations into the multiplier algebra $M(\mathcal{K}(L^2(G)))$. Then $(\mathrm{id} \otimes M) \circ \widetilde{\alpha}$ is a nondegenerate homomorphism of A to $M(A \otimes \mathcal{K}(L^2(G)))$. Similarly, $1_{M(A)} \otimes \lambda$ is a representation of G in $M(A \otimes \mathcal{K}(L^2(G)))$. It's an exercise to check that the pair $((\mathrm{id} \otimes M) \circ \widetilde{\alpha}, 1 \otimes \lambda)$ satisfies the covariance property. The *reduced crossed product* $A \rtimes_{\alpha,r} G$ is the image of $A \rtimes_\alpha G$ under the integrated form

$$\Lambda = \big((\mathrm{id} \otimes M) \circ \widetilde{\alpha}\big) \times (1 \otimes \lambda).$$

Theorem 4.15. *Λ is faithful, which we write as "$A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$", if G is amenable.*

Outline of proof. By amenability, λ is faithful on $C^*(G)$, and if $\pi \times U$ is a faithful representation of $A \rtimes_\alpha G$ it is not hard to deduce that

$$\text{“}(\pi \times U) \otimes \lambda\text{”} = (\pi \otimes 1) \times (U \otimes \lambda)$$

is faithful. But then some standard representation theory shows that this is equivalent to a regular representation of (A, α). □

Example 4.16. $C_r^*(G) = \mathbb{C} \rtimes_{\text{trivial action},r} G$, since the regular representation of the trivial action is really just the regular representation of G.

Theorem 4.17 (Stone-von Neumann Theorem)**.** $C_0(G) \rtimes_{\mathrm{lt}} G = \mathcal{K}(L^2(G))$

Outline of proof. As mentioned in Example 4.7, (M, λ) is a covariant representation of $(C_0(G), \mathrm{lt})$ on $L^2(G)$, and the integrated form $M \times \lambda$ takes the subalgebra $C_c(G, C_c(G)) \subseteq C_c(G, C_0(G))$ to the set of all kernel operators with kernel in $C_c(G \times G)$, which is a dense subset of $\mathcal{K}(L^2(G))$. The fact that this representation $M \times \lambda$ of $C_0(G) \rtimes_{\mathrm{lt}} G$ is faithful is deeper, and is usually proved using the theory of induced representations. □

Remark 4.18. An immediate consequence: $C_0(G) \rtimes_{\mathrm{lt}} G = C_0(G) \rtimes_{\mathrm{lt},r} G$. Thus, the converse of Theorem 4.15 is false.

5 Takai-Takesaki crossed-product duality

Throughout this section, G will be abelian.

Definition 5.1. Let (A, α) be an action of G. The *dual action* $\widehat{\alpha}$ of the dual group $\widehat{G}$ on $A \rtimes_\alpha G$ is given on the generators by

$$\widehat{\alpha}_\chi(i_A(a)) = i_A(a)$$
$$\widehat{\alpha}_\chi(i_G(s)) = \chi(s) i_G(s).$$

The justification that this actually gives an action is an exercise in the universal property of (i_A, i_G), together with continuity of uniform limits of continuous functions.

Theorem 5.2 (Takai-Takesaki duality)**.** *If (A, α) is an action of G, then*

$$A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G} \simeq A \otimes \mathcal{K}(L^2(G)).$$

Outline of proof. As Williams [Wil07, discussion following Theorem 7.1] indicates, Raeburn's argument [Rae88, Theorem 6] can be reformulated so that the strategy is to verify the steps in the following chain of isomorphisms:

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \xrightarrow{\simeq} (A \otimes C_0(G)) \rtimes_{\alpha \otimes \mathrm{rt}} G$$
$$\xrightarrow{\simeq} (A \otimes C_0(G)) \rtimes_{\mathrm{id} \otimes \mathrm{rt}} G \xrightarrow[\simeq]{} A \otimes (C_0(G) \rtimes_{\mathrm{rt}} G) \xrightarrow[\simeq]{} A \otimes \mathcal{K}(L^2(G)).$$

The point is that A is shifted around until it becomes a freely moving object. Of course, the Stone-von Neumann theorem $C_0(G) \rtimes G \simeq \mathcal{K}(L^2(G))$ is used at the last step (and the shift from lt to rt causes no harm). $\square$

It's possible to keep track of what the isomorphism does to the double-dual action of G, but we refrain from making this precise because the Takai-Takesaki duality theorem will be superseded by Imai-Takai duality (see Section 7), and at that time we'll take care of the double-dual action.

It's worth pointing out that although we outlined a strategy based upon Raeburn's paper from the late 1980's, the result itself dates from the 1970's. Takai's original proof [Tak75] was heavily representation-theoretic, and was based upon Takesaki's crossed-product duality theorem [Tak73] for von Neumann algebras.

6 Coactions

The Takai duality theorem is so useful that everyone wanted a version for nonabelian G. But then there is no dual group, and consequently no dual action. The fix involves a different sort of duality, which can ultimately be traced back to Fourier transforms: if G is abelian, then $C^*(G) \simeq C_0(\widehat{G})$ and this leads us to regard $C_0(G)$ and $C^*(G)$ as dual structures. An action (A, α) of G can be characterized in terms of the homomorphism

$$\widetilde{\alpha}\colon A \to M(A \otimes C_0(G))$$

we discussed earlier, and then replacing $C_0(G)$ by $C^*(G)$ gives rise to coactions of G. To prepare for the definition we have to note that one of the conditions on the above $\widetilde{\alpha}$ involves the *comultiplication* on $C_0(G)$, which is the homomorphism

$$\Delta_G\colon C_0(G) \to C_b(G \times G) = M(C_0(G) \otimes C_0(G))$$

given by

$$\Delta_G(f)(s,t) = f(st).$$

The dual version of this is:

Definition 6.1. The *comultiplication*

$$\delta_G\colon C^*(G) \to M(C^*(G) \otimes C^*(G))$$

is the integrated form of the representation given by

$$\delta_G(s) = s \otimes s.$$

We will use the comultiplication to define the dual version of an action. But first, for technical reasons it is convenient to introduce a special variant of the multiplier algebra:

Definition 6.2. Given two C^*-algebras A, B, we define

$$\widetilde{M}(A \otimes B) = \{m \in M(A \otimes B) : m(1 \otimes B) \cup (1 \otimes B)m \in A \otimes B\}.$$

Definition 6.3. A *coaction* (A, δ) of G is an injective nondegenerate homomorphism

$$\delta\colon A \to \widetilde{M}(A \otimes C^*(G))$$

such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta \tag{1}$$

and

$$\overline{\mathrm{span}}\{\delta(A)(1 \otimes C^*(G))\} = A \otimes C^*(G). \tag{2}$$

The condition (1) is a dual version of the homomorphism property of an action $\alpha\colon G \to \mathrm{Aut}\, A$. The last condition (2), which makes sense because δ maps into $\widetilde{M}(A \otimes C^*(G))$, is called *nondegeneracy* of the coaction, and is crucial for duality theory. It remains an open problem whether it is redundant. In the older literature the definition of coaction omitted this condition, but then every result appealing to duality had to refer to a "nondegenerate coaction".

Example 6.4. If G is abelian, a Fourier-transform argument shows that a coaction of G is just a different way of looking at an action of $\widehat{G}$. Dually, an action of G may be considered as a coaction of $\widehat{G}$.

Example 6.5. The *trivial coaction* on any A is given by

$$a \mapsto a \otimes 1\colon A \to \widetilde{M}(A \otimes C^*(G)).$$

In order to define covariant representations of coactions, we need an auxiliary object w_G defined below. But first we need the following fact concerning multipliers and tensor products:

Lemma 6.6. *For any C^*-algebra A and locally compact Hausdorff space X,*

$$M(A \otimes C_0(X)) \simeq M(C_0(X) \otimes A) \simeq C_b(X, M^\beta(A)),$$

where the notation β indicates that we require continuity in the strict topology of $M(A)$. Moreover,

$$\widetilde{M}(A \otimes C_0(X)) \simeq C_b(X, A).$$

Of course the underlying facts in the above lemma are the canonical isomorphisms

$$A \otimes C_0(X) \simeq C_0(X) \otimes A \simeq C_0(X, A).$$

Definition 6.7. The *canonical unitary* w_G is the element of $M(C_0(G) \otimes C^*(G)) = C_b(G, M^\beta(C^*(G)))$ given by the canonical embedding $G \to M(C^*(G))$:

$$w_G(s) = s \quad \text{for } s \in G.$$

Remark 6.8. Of course, w_G is just another name for the embedding $i_G \colon G \to M(C^*(G))$ that I used when introducing $C^*(G)$. But the emphasis here is on the particular connection with tensor products, and it helps keep track of things to use a new name.

Definition 6.9. A *covariant representation* $(\pi, \mu) \colon (A, \delta) \to M(B)$ is a pair of nondegenerate homomorphisms

$$A \xrightarrow{\pi} M(B) \xleftarrow{\mu} C_0(G)$$

such that the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\quad\delta\quad} & M(A \otimes C^*(G)) \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi\otimes\mathrm{id}} \\
M(B) & \xrightarrow[\mathrm{Ad}(\mu\otimes\mathrm{id})(w_G)\circ(\cdot\otimes 1)]{} & M(B \otimes C^*(G))
\end{array}$$

commutes.

Remark 6.10. To parse the bottom arrow, note that since μ is nondegenerate so is

$$\mu \otimes \mathrm{id} \colon C_0(G) \otimes C^*(G) \to M(B \otimes C^*(G)).$$

Thus we can extend uniquely to the multiplier algebra $M(C_0(G) \otimes C^*(G))$ to get a unitary element

$$(\mu \otimes \mathrm{id})(w_G) \in M(B \otimes C^*(G)).$$

Thus for $b \in B$ we can conjugate $b \otimes 1_{M(C^*(G))}$:

$$\mathrm{Ad}(\mu \otimes \mathrm{id})(w_G)(b \otimes 1) \in M(B \otimes C^*(G)).$$

This gives a nondegenerate homomorphism

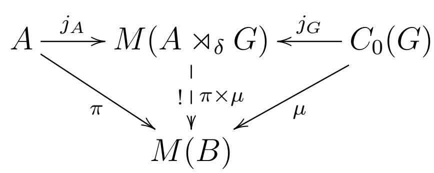
$$\mathrm{Ad}(\mu \otimes \mathrm{id}(w_G)) \circ (\cdot \otimes 1)\colon B \to M(B \otimes C^*(G)).$$

By nondegeneracy, this map extends uniquely to $M(B)$, so composing with π makes sense.

Remark 6.11. Again, when G is abelian, a Fourier-transform argument shows that covariant representations of a coaction of G are just a different way of looking at covariant representations of the associated action of $\widehat{G}$.

Example 6.12. The *regular representation* of (A, δ) is constructed as follows: Again regard both the integrated form of the regular representation λ of G and the representation M of $C_0(G)$ as representations into $M(\mathcal{K}(L^2(G)))$. Then $(\mathrm{id}_A \otimes \lambda) \circ \delta$ and $1_{M(A)} \otimes M$ are representations of A and $C_0(G)$ in $M(A \otimes \mathcal{K}(L^2(G)))$, and the pair $((\mathrm{id} \otimes \lambda) \circ \delta, 1 \otimes M)$ is a covariant representation, called the *regular representation*, of (A, δ) in $M(A \otimes \mathcal{K}(L^2(G)))$. Observe that the λ and M are reversed from their positions in the regular representation of an action; this is just another instance of the $C^*(G) - C_0(G)$ duality.

Definition 6.13. A *crossed product* of (A, δ) is a C^*-algebra $A \rtimes_\delta G$ together with a covariant representation $(j_A, j_G)\colon (A, \delta) \to M(A \rtimes_\delta G)$ such that for every covariant representation $(\pi, \mu)\colon (A, \delta) \to M(B)$ there is a unique nondegenerate homomorphism $\pi \times \mu\colon A \rtimes_\delta G \to M(B)$, called the *integrated form* of (π, μ), such that the diagram



commutes.

Theorem 6.14. *$A \rtimes_\delta G$ exists and is unique up to isomorphism.*

Outline of proof. In fact we can get away with just using the regular representation:

$$j_A = (\mathrm{id} \otimes \lambda) \circ \delta$$
$$j_G = 1 \otimes M$$

$$A \rtimes_\delta G = C^*\big(j_A(A)j_G(C_0(G))\big).$$

This works basically because a coaction is analogous to an action of an abelian group, which is amenable. I'll just mention that it is a bit of a trick to verify that every covariant representation factors through the regular one. This gives existence, and uniqueness follows (as usual) from the universal property. □

Remark 6.15. Note that, unlike for actions, this time there is no convenient choice of dense *-subalgebra like $C_c(G, A)$.

Definition 6.16. Given two coactions (A, δ) and (B, ε), a homomorphism $\pi\colon A \to B$ is $\delta-\varepsilon$ *equivariant* if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\delta\quad} & \widetilde{M}(A \otimes C^*(G)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi\otimes\mathrm{id}} \\ B & \xrightarrow[\quad\varepsilon\quad]{} & M(B \otimes C^*(G)) \end{array}$$

commutes. An *isomorphism* $\pi\colon (A, \delta) \to (B, \varepsilon)$ is an equivariant homomorphism that is also an isomorphism of A onto B. If such an isomorphism exists, we say that the coactions $(A, \delta), (B, \varepsilon)$ are *isomorphic*, written $(A, \delta) \simeq (B, \varepsilon)$.

Remark 6.17. There is subtlety in the above diagram: Since $\pi\colon A \to B$ may be degenerate, in which case $\pi \otimes \mathrm{id}\colon A \otimes C^*(G) \to B \otimes C^*(G)$ will be degenerate as well, so we can't expect to extend it to

$$M(A \otimes C^*(G)) \to M(B \otimes C^*(G)).$$

However, by [EKQR06, Proposition A.6] there is a canonical extension

$$\pi \otimes \mathrm{id}\colon \widetilde{M}(A \otimes C^*(G)) \to M(B \otimes C^*(G)).$$

In fact, the image is in $\widetilde{M}(B \otimes C^*(G))$, although we did not need to know that.

Example 6.18. If $\pi\colon A \to B$ is $\delta - \varepsilon$ equivariant, then

$$(j_B \circ \pi, j_G^\varepsilon)\colon (A, \delta) \to M(B \rtimes_\varepsilon G)$$

is a covariant homomorphism, and the integrated form maps into the crossed product:

$$\pi \rtimes G\colon A \rtimes_\delta G \to B \rtimes_\varepsilon G.$$

Remark 6.19. In the abelian case, equivariant homomorphisms for coactions of G correspond bijectively to equivariant homomorphisms for actions of the dual group $\widehat{G}$.

Example 6.20. $C_0(G) = \mathbb{C} \rtimes_{\text{trivial coaction}} G$. This a special case of the following example.

Example 6.21. $A \rtimes_{\text{trivial coaction}} G \simeq A \otimes C_0(G)$, since covariant representations are just commuting homomorphisms, and $C_0(G)$ is nuclear, so the maximal and minimal tensor products coincide.

Example 6.22. $C^*(G) \rtimes_{\delta_G} G \simeq \mathcal{K}(L^2(G))$. This is the dual version of the Stone-von Neumann Theorem, and the parallel between them is quite strong: a pair (U, μ) is covariant for the coaction $(C^*(G), \delta_G)$ if and only if the pair (μ, U) is covariant for the action $(C_0(G), \mathrm{lt})$.

7 Imai-Takai duality

Using coactions, the Takai duality theorem can be extended to nonabelian groups.

Definition 7.1. If (A, α) is an action of G, the *dual coaction* $\widehat{\alpha}$ of G on the crossed product $A \rtimes_\alpha G$ is determined by the covariant representation

$$\begin{aligned} a &\mapsto i_A(a) \otimes 1 \quad \text{for } a \in A \\ s &\mapsto i_G(s) \otimes s \quad \text{for } s \in G \end{aligned}$$

of the action (A, α) in $M((A \rtimes_\alpha G) \otimes C^*(G))$.

Example 7.2. The dual of the trivial action of G on $\mathbb{C}$ is the coaction δ_G on $C^*(G)$.

Example 7.3. If $\pi\colon A \to B$ is $\alpha - \beta$ equivariant, then $\pi \rtimes G\colon A \rtimes_\alpha G \to B \rtimes_\beta G$ is $\widehat{\alpha} - \widehat{\beta}$ equivariant.

Definition 7.4. If (A, δ) is a coaction of G, the *dual action* $\widehat{\delta}$ of G on the crossed product $A \rtimes_\delta G$ is defined as follows: for $s \in G$ the automorphism $\widehat{\delta}_s$ of $A \rtimes_\delta G$ is determined by the covariant representation

$$\begin{aligned} a &\mapsto j_A(a) \quad \text{for } a \in A \\ f &\mapsto j_G(\mathrm{rt}_s(f)) \quad \text{for } f \in C_0(G) \end{aligned}$$

of the coaction (A, δ) in $M(A \rtimes_\delta G)$.

Example 7.5. The dual of the trivial coaction of G on $\mathbb{C}$ is the action rt on $C_0(G)$.

Example 7.6. If $\pi\colon A \to B$ is $\delta - \varepsilon$ equivariant, then $\pi \rtimes G\colon A \rtimes_\delta G \to B \rtimes_\varepsilon G$ is $\widehat{\delta} - \widehat{\varepsilon}$ equivariant.

Theorem 7.7 (Imai-Takai duality)**.** *If (A, α) is an action of G, then*

$$\left(A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} G, \widehat{\widehat{\alpha}}\right) \simeq \left(A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \operatorname{Ad}\rho\right).$$

Outline of proof. The argument in [Rae87, Theorem 7] can be reformulated similarly to the abelian case (Theorem 5.2), which itself depends upon the isomorphism $C^*(\widehat{G}) \simeq C_0(G)$. □

This result was proved using heavily representation-theoretic techniques in [IT78, Theorem 3.6].

8 Katayama crossed-product duality

Symmetrically, we should have a duality starting with coactions.

Definition 8.1. The *canonical surjection* $\Phi\colon A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \to A \otimes \mathcal{K}(L^2(G))$ is determined by the covariant representation $(\Lambda, 1 \otimes \rho)$ of the dual action $(A \rtimes_\delta G, \widehat{\delta})$, where Λ is the regular representation of (A, δ).

Definition 8.2. A coaction δ is called *maximal* if Φ is injective, and hence an isomorphism.

Example 8.3. Every dual coaction $\widehat{\alpha}$ is maximal.

Theorem 8.4 (Maximal Katayama Duality)**.** *If (A, δ) is a maximal coaction of G, then*

$$\left(A \rtimes_\delta G \rtimes_{\widehat{\delta}} G, \widehat{\widehat{\delta}}\right) \simeq \left(A \otimes \mathcal{K}(L^2(G)), \operatorname{Ad}(1 \otimes W) \circ (\delta \otimes_* id)\right)$$

where

$$W = (M \otimes id)(w_G^*)$$

and $\delta \otimes_* id$ is the coaction of G on $A \otimes \mathcal{K}(L^2(G))$ given by the composition

$$\begin{array}{ccc} A\otimes\mathcal{K}(L^2(G)) & \xrightarrow{\delta\otimes id} & M(A\otimes C^*(G)\otimes\mathcal{K}(L^2(G))) \\ & \searrow^{\delta\otimes_* id} & \downarrow{\scriptstyle id\otimes\Sigma} \\ & & M(A\otimes\mathcal{K}(L^2(G))\otimes C^*(G)), \end{array}$$

where in turn $\Sigma : C^*(G) \otimes \mathcal{K}(L^2(G)) \to \mathcal{K}(L^2(G)) \otimes C^*(G)$ is the isomorphism determined by

$$c \otimes k \mapsto k \otimes c.$$

Proof. The strategy is as before: rearrange A to decouple it from G, and get $\mathcal{K}(L^2(G))$ from the representations of $C_0(G)$ and $C^*(G)$. □

Example 8.5. The special case $A = \mathbb{C}$ is the Stone-von Neumann Theorem, because rt is the dual action of the trivial coaction. However, this doesn't really give an independent proof of Stone-von Neumann.

Definition 8.6. A coaction δ is *normal* if the canonical homomorphism $j_A \colon A \to M(A \rtimes_\delta G)$ is faithful.

Example 8.7. The regular representation $\Lambda \colon A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ is equivariant for $\widehat{\alpha}$ and a normal coaction $\widehat{\alpha}^r$.

Theorem 8.8 (Normal Katayama Duality)**.** *If (A, δ) is a normal coaction of G, then there is a commutative diagram*

$$\begin{array}{ccc} A\rtimes_\delta G\rtimes_{\widehat{\delta}} G & \xrightarrow{\Phi} & A\otimes\mathcal{K}(L^2(G)) \\ {\scriptstyle\Lambda}\downarrow & \nearrow_{\simeq} & \\ A\rtimes_\delta G\rtimes_{\widehat{\delta},r} G & & \end{array}$$

In fact, a coaction is normal if and only if the canonical surjection Φ factors through the regular representation of the dual action [EKQ04, Proposition 2.2].

Definition 8.9. Let (A, δ) be a coaction.

(1) A *maximalization* of δ is a maximal coaction (A^m, δ^m) and a $\delta^m - \delta$ equivariant surjection $\psi\colon A^m \to A$ such that

$$\psi \rtimes G\colon A^m \rtimes_{\delta^m} G \to A \rtimes_\delta G$$

is an isomorphism.

(2) A *normalization* of δ is a normal coaction (A^n, δ^n) and a $\delta - \delta^n$ equivariant surjection $\eta\colon A \to A^n$ such that

$$\eta \rtimes G\colon A \rtimes_\delta G \to A^n \rtimes_{\delta^n} G$$

is an isomorphism.

Theorem 8.10. *Maximalizations and normalizations always exist, and are unique up to isomorphism. Moreover, every coaction (A, δ) fits into a commutative diagram*

$$\begin{array}{ccc} (A^m, \delta^m) & & \\ & \overset{\psi}{\searrow} & \\ \Lambda\downarrow & & (A, \delta) \\ & \underset{\eta}{\swarrow} & \\ (A^n, \delta^n). & & \end{array}$$

Example 8.11. $\Lambda\colon (A \rtimes_\alpha G, \widehat{\alpha}) \to (A \rtimes_{\alpha,r} G, \widehat{\alpha}^r)$ is both a normalization of $\widehat{\alpha}$ and a maximalization of $\widehat{\alpha}^r$.

9 Landstad duality

Imai-Takai duality allows recovery of an action of G, up to Morita equivalence, from the dual coaction (and similarly by Katayama duality we can recover a maximal or normal coaction from its dual action). But if we are given just a bit more data we can recover the original up to isomorphism (and who could ask for anything more?).

Theorem 9.1 (Landstad duality)**.** *Let (B, δ) be a maximal coaction of G, and let $U\colon G \to M(B)$ be a representation such that*

$$\delta(U_s) = U_s \otimes s \quad \text{for } s \in G.$$

Then there is an action (A, α), unique up to isomorphism, such that

$$(A \rtimes_\alpha G, \widehat{\alpha}, i_G) \simeq (B, \delta, U).$$

In other words, there is an $\widehat{\alpha} - \delta$ equivariant isomorphism $A \rtimes_\alpha G \to B$ taking i_G to U. One way to interpret Landstad duality is that, given not only the dual coaction $(A \rtimes_\alpha G, \widehat{\alpha})$, but also the representation i_G, we can recover the action not just up to Morita equivalence but up to isomorphism.

Remark 9.2. The proof is quite hard, even when G is abelian (in which case coactions are not needed); in [Ped79, Theorem 7.8.8], Pedersen takes 1.5 pages to prove the abelian version, and he needs a proposition and three lemmas to prepare — this is quite a lot for him! The hard part is constructing a suitable A from the data, and involves a delicate averaging process. Landstad's proof [Lan79, Theorem 3] uses the reduced crossed product $A \rtimes_{\alpha,r} G$ and what are nowadays called “reduced coactions”, which are more-or-less the same as normal coactions [Rae87, Rae92, Qui94], but use $C^*_r(G)$ rather than $C^*(G)$. Theorem 9.1 is a version for the full crossed product and maximal coaction, and appears in [KQ07, Theorem 3.2].

There is a dual version, recovering coactions (instead of actions) up to isomorphism [KQR08, Theorem 4.2 and Corollary 4.3].

10 “Fixing” the Baum-Connes Conjecture

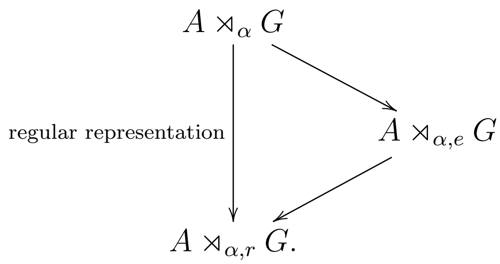
The Baum-Connes Conjecture says that the K-theory of the reduced crossed product is naturally isomorphic to a “topological K-theory” — we won't discuss either of these directly. It transpires that the conjecture is false in its original form, because the topological K-theory is an exact functor of actions, while the reduced crossed product is not, thanks to Gromov's discovery of nonexact groups.

Baum wants to fix the conjecture, so together with Guentner and Willett [BGW16] he searched for an alternative version of the crossed product that would do the job. Although the full crossed product is an exact functor, it is too big, and leads to failure of the Baum-Connes conjecture for different reasons. So, Baum-Guentner-Willett initiated the study of *exotic crossed products*, lying between the full and reduced ones. The first hurdle is finding examples of these, and the hard part is that it must be functorial, and it's certainly required to be exact. Perhaps in an attempt to stay as far away as possible from the sort of counterexamples involving full crossed products, the focus is on the *minimal exact crossed-product functor* (meaning it should be as close as possible to the reduced crossed product). This exists uniquely, by abstract nonsense. So the problem is, how to find it?

Let's take a closer look: An *exotic crossed product*

$$(A, \alpha) \mapsto A \rtimes_{\alpha,e} G$$

is a functor from actions to C^*-algebras that fits into a commutative diagram of natural surjections:



It's no accident that this looks a lot like the diagram in Theorem 8.10. Although [BGW16] does not mention coactions, it occurred to us [KLQ13, KLQ16b, KLQ16a, KLQ18] that since the full and reduced crossed products have dual coactions, we could recast the theory of exotic crossed products by requiring that $A \rtimes_{\alpha,e} G$ carry a version of the dual coaction, and that it be computed by first taking the full crossed product and then applying a *coaction functor*.

Example 10.1. Taking full crossed product and then applying the normalization functor reproduces the reduced-crossed-product.

I'll close by describing two sources of coaction functors.

KLQ functors

The first construction begins with the group algebra: consider a quotient map $q_E\colon C^*(G) \to C^*_E(G)$ such that:

- the regular representation λ factors through q_E, and
- q_E is equivariant for δ_G and a coaction δ_G^E.

Then for any coaction (A, δ) consider the composition

$$\begin{array}{ccc} A & \xrightarrow{\delta} & \widetilde{M}(A \otimes C^*(G)) \\ & \searrow^{Q^E} & \downarrow{\scriptstyle \mathrm{id} \otimes q_E} \\ & & \widetilde{M}(A \otimes C^*_E(G)). \end{array}$$

The image $A^E = Q^E(A)$ carries a quotient δ^E of the coaction δ, and the assignments $(A, \delta) \mapsto (A^E, \delta^E)$ give a coaction functor τ_E that we studied in [KLQ16a, KLQ18]. Unfortunately, it seems quite hard to find nontrivial examples of these functors that are exact. Even worse, it transpires that the minimal exact crossed product cannot be of the form τ_E [BEWb, discussion following Corollary 4.7]. As a result, our interest in KLQ coaction functors has diminished.

Tensor D functors

Now we generalize somewhat the preceding construction. Replace the quotient map $q_E \colon C^*(G) \to C^*_E(G)$ by a homomorphism $V \colon C^*(G) \to D$ that is equivariant for δ_G and a coaction ζ. For this construction we also need to modify our coactions using the maximal tensor product rather than the minimal one: consider the composition

$$\begin{array}{ccc} A & \xrightarrow{\delta} & \widetilde{M}(A \otimes_{\max} C^*(G)) \\ & \searrow^{Q^D} & \downarrow{\scriptstyle \mathrm{id} \otimes V} \\ & & \widetilde{M}(A \otimes_{\max} D). \end{array}$$

Then again the image $A^D = Q^D(A)$ carries a quotient δ^D of δ, and this gives a coaction functor τ^D.

It follows from Landstad duality that the coaction (D, ζ) is of the form $(C \rtimes_\gamma G, \widehat{\gamma})$ for some action γ of G on a unital C^*-algebra C. We are still working out the details, but we are 95% sure that the functor τ^D is exact, and composing τ^D with the full crossed product recovers the "C-crossed product"

$$(B, \alpha) \mapsto B \rtimes_{\alpha, C} G$$

introduced in [BGW16]. We know that the minimal such coaction functor reproduces the minimal C-crossed product studied in [BEW18]. It's worth mentioning that for all anyone knows the minimal C-crossed product might in fact be the minimal exact crossed product.

While not all crossed-product functors arise via coaction functors as above, all the "good" ones — in this case meaning all those that behave well with C^*-correspondences — do. Our ultimate goal is to derive all the important facts about exotic crossed products from within the world of coaction functors. We have made significant progress, but much work still remains.

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