Synchronous games and game algebras

Let \( \mathcal{G} = (X, X, A, A, \lambda) \) be a synchronous game, i.e.

\[
\lambda(x, x, a, b) = 0 \text{ if } a \neq b,
\]

(e.g. \( \text{Hom}(\mathcal{G}, \mathcal{H}) \)) then winning strategies are characterised by traces on

\[
\mathcal{A}(\lambda) = \mathcal{A}_{X, A}/J(\lambda) \quad \text{(the game algebra)}
\]

with \( J(\lambda) = \langle e_x, a e_y, b : \lambda(x, y, a, b) = 0 \rangle \).

We have (Helton-Meyer-Paulsen-Satriano,Kim-Paulsen-Schafhauser):

- \( \mathcal{G} \) has a perfect \( C_{qc} \) strategy iff there exists a non-zero tracial \( C^* \)-algebra \((\mathcal{A}, \tau)\) and a \(*\)-homomorphism \( \pi : \mathcal{A}(\lambda) \to \mathcal{A} \). Then

\[
p(a, b|x, y) = \tau(e_x, a e_y, b),
\]

- \( C_{qa} \) strategy iff we can take \( \mathcal{A} = \mathcal{R}^U \);
- \( C_q \) strategy iff we can take \( \mathcal{A} \) finite-dimensional;
- \( C_{loc} \) strategy iff we can take \( \mathcal{A} \) abelian.
Homomorphism and isomorphism games

For $t \in \{loc, q, qa, qc\}$ and graphs $G$ and $H$, write $G \rightarrow_t H$ if there exists a perfect $C_t$ strategy for the homomorphism game $\text{Hom}(G, H)$. One can define quantum analogues of chromatic numbers and independence numbers $\chi(G)$ and $\alpha(G)$:

- The $t$-chromatic number of a graph $G$ is
  $$\chi_t(G) = \min\{d : G \rightarrow_t K_d\}$$

- The $t$-independence number of $G$ is
  $$\alpha_t(G) = \max\{d : K_d \rightarrow_t \bar{G}\}.$$ 

- $\chi(G) = \chi_{loc}(G) \geq \chi_q(G) \geq \chi_{qa}(G) \geq \chi_{qc}(G)$,
- $\alpha(G) = \alpha_{loc}(G) \leq \alpha_q(G) \leq \alpha_{qa}(G) \leq \alpha_{qc}(G)$. 

L. Turowska
Quantum no-signalling correlations and quantum non-local games
Hom}(G, H) games are rich enough to witness differences between $C_q$, $C_{qa}$ and $C_{qc}$:


$$\exists G : \chi_q(G) > \chi_{qa}(G)$$

Mancinska-Roberson-Varvitsiotis (2020) using MIP$^* = \text{RE}$ (2020):

$$\exists G : \alpha_{qc}(G) > \alpha_{qa}(G)$$

Iso}(G, H) games are examples of bisynchronous games where

$$\lambda(x, x, a, b) = 0 \text{ and } \lambda(x, y, a, a) = 0 \text{ if } x \neq y, a \neq b.$$ 

Perfect strategies are captured by quantum permutation group: $O(S_X^+)$ is generated by $p_{a,x} = p_{a,x}^* = p_{a,x}^2$ s.t. $U = (p_{a,x})_{a,x \in X}$ is a magic unitary, i.e. $\sum_a p_{a,x} = \sum_y p_{b,y} = 1, x, b \in X$. 

L. Turowska

Quantum no-signalling correlations and quantum non-local games
If \( A_G \) and \( A_H \) are the adjacency matrices of \( G \) and \( H \) resp. then

\[
\mathcal{A}(\text{Iso}(G, H)) = \langle p_{a,x} : U = (p_{a,x})_{a,x} \text{ magic unitary with } (A_G \otimes 1)U = U(A_H \otimes 1) \rangle.
\]

Combining the previous results one gets:

For \( \text{Iso}(G,H) \) we have:

- \( G \cong_{qc} H \) iff there is a trace on \( \mathcal{A}(\text{Iso}(G, H)) \)
- \( G \cong_{q} H \) iff there is a fin.-dim. repr. of \( \mathcal{A}(\text{Iso}(G, H)) \)
- \( G \cong_{loc} H \) iff \( G \cong H \) iff there is a one-dim repr. of \( \mathcal{A}(\text{Iso}(G, H)) \).

\[
\begin{align*}
G \cong H & \not\Rightarrow \quad G \cong_{q} H \not\Rightarrow \quad G \cong_{qa} H \Rightarrow \quad G \cong_{qc} H \\
G \cong_{qc} H & \iff G \cong_{C^*} H \iff G \cong_{alg} H \quad (\text{[BCEHPSW]})
\end{align*}
\]
Motivating question I: Suppose the game has quantum inputs/outputs. What kind of strategies can be used?

Motivating question II: Perhaps a suitable (simpler and genuinely) quantum game can disprove Tsirelson-Connes?

A classical input \((x, y) \mapsto \text{state } \epsilon_{x,x} \otimes \epsilon_{y,y} \in \mathcal{D}_X \otimes \mathcal{D}_Y\) (pos.-semidef. matrix of trace 1) here matrix units: \(\epsilon_{x,y}, x, y \in X\) in \(M_X\).

A correlation \(p = \{(p(a, b|x, y)_{a,b} : x, y \in X\} \mapsto \mathcal{N}_p : \mathcal{D}_{X \times Y} \rightarrow \mathcal{D}_{A \times B},\)

\[
\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,b \in A} p(a, b|x, y)\epsilon_{a,a} \otimes \epsilon_{b,b}.
\]

Note: \(p\) is no-signalling \(\iff\)

\[
\text{Tr}_A \mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \text{Tr}_A \mathcal{N}_p(\epsilon_{x',x'} \otimes \epsilon_{y,y}) \quad \text{and} \quad \text{Tr}_B \mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \text{Tr}_B \mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y',y'})
\]

Quantisation: (Duan-Winter) Quantum channels (completely positive trace preserving)

\[
\Gamma : M_{XY} \rightarrow M_{AB}, \text{ satisfying no-signalling conditions.}
\]

No-signalling:

\[
\text{Tr}_A \Gamma(\omega_X \otimes \omega_Y) = \text{Tr}_A \Gamma(\omega'_X \otimes \omega_Y) \quad \text{and} \quad \text{Tr}_B \Gamma(\omega_X \otimes \omega_Y) = \text{Tr}_B \Gamma(\omega_X \otimes \omega'_Y)
\]
Classes of quantum no-signalling correlations (Todorov-T. (2020))

A family of classical POVM’s:
\[
\{(E_x,a)_{a \in A} : x \in X\}
\]
\[\leadsto\]
\[
E = \sum_{x \in A} \sum_{a \in A} \epsilon_{x,a} \otimes E_{x,a} \in M_{XA}(B(H))^+\]

A family of quantum POVM’s:
Stochastic operator matrix
\[
E = (E_{x,x',a,a'}) \in M_{XA}(B(H))^+ \text{ such that } \operatorname{Tr}_A E = I \otimes I_X.
\]

Classes of quantum no-signalling (QNS) correlations

\[
\uparrow
\]

Quantum channels \( \Gamma : M_{XY} \to M_{AB} \) with specified Choi matrices
\[
(\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}))_{x,x',y,y'} \in M_{XYAB}
\]

Local QNS

Convex combinations of \( \Phi \otimes \Psi \)

Quantum QNS

\[
\langle E_{x,x',a,a'} \otimes F_{y,y',b,b'} \xi, \xi \rangle,
\]
\( \xi \in H_A \otimes H_B \)

Quantum commuting QNS

\[
\langle E_{x,x',a,a'} F_{y,y',b,b'} \xi, \xi \rangle,
\]
\( \xi \in H \)

\( \mathcal{Q}_{loc} \subset \mathcal{Q}_q \subset \mathcal{Q}_{qa} \subset \mathcal{Q}_{qc} \subset \mathcal{Q}_{ns} \)
Quantum non-local games

Classical games \((X, Y, A, B, \lambda)\):

Rule function \(\lambda : X \times Y \times A \times B \to \{0, 1\} \sim (P(x,y), P_{\beta,x,y}(\lambda))\)

\(\beta_{x,y}(\lambda) = \{(a, b) : \lambda(x, y, a, b) = 1\}\),

\(P_\alpha\) is a projection onto \(\text{span}\{e_x \otimes e_y : (x, y) \in \alpha\}\)

\(\lambda \sim \varphi_\lambda : \text{Proj}(\mathcal{D}_{XY}) \to \text{Proj}(\mathcal{D}_{AB})\), \(\lor\)-preserving 0-preserving map.

Quantum games:

Replace \((X, Y, A, B, \lambda)\) by \(\text{Proj}(M_{XY}), \text{Proj}(M_{AB})\) and

\(\varphi : \text{Proj}(M_{XY}) \to \text{Proj}(M_{AB})\), \(\lor\)-preserving 0-preserving map.

Definition (Todorov-T, 2020)

A QNS correlation \(\Gamma : M_{XY} \to M_{AB}\) is a perfect strategy for the quantum non-local game \(\varphi\) if 

\[\langle \Gamma(P), \varphi(P)\bot \rangle := \text{Tr}(\Gamma(P)\varphi(P)\bot) = 0, \, \forall P \in \text{Proj}(M_{XY}).\]
Questions and some references

"Synchronous" and "bisynchronous" quantum games and associated algebras.

Quantum graph homomorphisms and isomorphisms and associated algebras.
Brannan-Chirvasitu-Eifler-Harris-Paulsen-Su-Wasilewski 2020,

Value of quantum non-local games.
Cooney-Junge-Palazuelos-Pérez-García 2015, Crann-Levene-Todorov-T-Winter, in progress