Supercurrents, minimal manifolds and mean curvature flow.

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A *superform* on $\mathbb{R}^n$ is a form on $\mathbb{C}^n$

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The complex structure on $\mathbb{C}^n$, J, acts on superforms. If a is of bidegree $(p, 0)$ we sometimes write $J(a) = a^\#$. If $J(a) = a$, a is symmetric, $a_{I,J} = a_{J,I}$.

We also define positivity for symmetric (p, p) forms:

$$a \geq 0$$

if

$$a \wedge \alpha_1 \wedge \alpha_1^\# \wedge ... \alpha_m \wedge \alpha_m^\# \geq 0.$$

Here $m = n - p$, α_j are $(1, 0)$.

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The superintegral does not depend on the orientation of $\mathbb{R}^n$, but it does depend on a choice of scalar product on $\mathbb{R}^n$.

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For instance (following Bedford-Taylor) we can define

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for ϕ convex and not necessarily smooth. It is the Alexandrov Monge-Ampère measure of ϕ.

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Let V be a hyperplane in $\mathbb{R}^n$. Its complexification V_s is a complex hyperplane in $\mathbb{R}^n_s$ so we can (super)integrate $(n-1, n-1)$-forms over V_s.

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A subspace of codim p defines a supercurrent in the same way

$$[V]_s = c_p[V] \wedge n_1^{\#} \wedge ...n_p^{\#}.$$

Submanifolds

If M is a submanifold of $\mathbb{R}^n$ of dimension m and codimension $p = n - m$, we define its associated supercurrent by

$$[M]_s = c_p[M] \wedge n_1^{\#} \wedge ... n_p^{\#} = (*dS_M) n_1 \wedge n_1^{\#} \wedge ... n_p \wedge n_p^{\#},$$

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Minimal submanifolds

A small computation gives

$$d[M]_s \wedge \beta^{n-2}/(n-2)! = tr(F)n^{\#}\rfloor[M]_s \wedge \beta^{n-1}/(n-1)!.$$

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with $F_j = dn_j^{\#}$ and $\sum tr(F_j)\vec{n}_j$ is again the mean curvature vector. So, $[M]_s \wedge \beta^{m-1}$ is closed precisely when M is minimal.

Note that $S := [M]_s \wedge \beta^{m-1}/(m-1)!$ is of bidegree $(n-1, n-1)$, i e of bidimension $(1, 1)$. But it is not an arbitrary $(n-1, n-1)$-current; it has the form $S = A \wedge \beta^{m-1}$, where $A \geq 0$.

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Now assume that M is minimal, so that S is closed. If u is a function

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This defines a Laplace operator on M which has no first order or second order terms, just like on a complex manifold. One verifies that the Newton kernel

$$E_{m-2} := -(1/(m-2))\frac{1}{|x|^{m-2}}$$

is subharmonic on $[M]_s$.

Volume computation à la Lelong

We look at the volume of M intersected with a ball of radius r

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From this we get the monotonicity theorem; $\sigma(r)/r^m$ is increasing. We also get that the Laplacian of E_{m-2} on M contains a point mass at the origin.

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$$|M| \geq \omega_m w(a).$$

As a consequence we get a result by Alexander-Osserman and Brendle-Hung:

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Let a be a point in the unit ball. Let M be an m-dimensional minimal manifold in the ball that contains a. Then

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So we can choose $w(x) = (1 + |a|^2 - 2a \cdot x)^{m/2}$, $w(a) = (1 - |a|^2)^{m/2}$.

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$$dS = -[M] \wedge F.$$

Thus $d^\# dS$ does not have measure coefficients, which looks bad. But it turns out that $dd^\# S$ has a nice interpretation in terms of the mean curvature flow.

Recall that $\vec{H} := \sum tr(F_j)\vec{n}_j$ ($F_j = dn_j^\#$) is the mean curvature vector field. It does not depend on the choice of ON-basis n_j.

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The flow exists for short times, but always collapses in finite time. (Look at a sphere.)

Recall that

$$dS = \vec{H}^{\#} \rfloor [M]_s \wedge \beta^m / m!.$$

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By Cartan's formula, this is the Lie derivative of σ along the flow (since $d\sigma = 0$). Keeping track of signs etc we get

Theorem

Let M_t be moving under the mean curvature flow. Then

$$\frac{d}{dt}[M_t]_s \wedge \beta^m/m! = -|\vec{H}|^2[M_t]_s \wedge \beta^m/m! - dd^{\#}S.$$

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decreases. As a consequence, if M_0 is contained in a convex set, M_t stays there.

Thanks!