Supercurrents, minimal manifolds and mean curvature flow.

Bo Berndtsson
Chalmers University of Technology
The aim of the talk is to introduce a formalism to study real submanifolds of $\mathbb{R}^n$ through methods that imitate complex analysis.

It is an elaboration of the work of Lagerberg, who applied similar techniques to tropical geometry.

We start with $\mathbb{R}^n = \{ x = (x_1, \ldots, x_n) \}$ and its complexification $\mathbb{C}^n = \{ x + i \xi = (x_1 + i \xi_1, \ldots, x_n + i \xi_n) \} =: \mathbb{R}^n_s$. We will think of $\mathbb{C}^n$ as the superspace of $\mathbb{R}^n$. A superform on $\mathbb{R}^n$ is a form on $\mathbb{C}^n$ $a = \sum a_{I, J}(x) \, dx^I \wedge d\xi^J$, where the coefficients $a_{I, J}$ do not depend on $\xi$.

If $|I| = p$ and $|J| = q$ we say that $a$ has bidegree $(p, q)$. The complex structure on $\mathbb{C}^n$, $J$, acts on superforms. If $a$ is of bidegree $(p, 0)$ we sometimes write $J(a) = a^\#$. If $J(a) = a$, $a$ is symmetric, $a_{I, J} = a_{J, I}$. 
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We also define positivity for symmetric \((p, p)\) forms:

\[ a \geq 0 \]

if

\[ a \wedge \alpha_1 \wedge \alpha_1^\# \wedge ... \alpha_m \wedge \alpha_m^\# \geq 0. \]

Here \(m = n - p\), \(\alpha_j\) are \((1, 0)\).
Integration

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\int_{\mathbb{R}^n_s} a := \int_{\mathbb{R}^n} a_0 dx \int d\xi,
\]

where \( \int d\xi := c_n = (-1)^n (n+1)/2 \) (if \( \xi_j \) are oriented and orthonormal). This is essentially the Berezin integral; the constant \( c_n \) is chosen so that \( \int a_0 dx_1 \wedge d\xi_1 \ldots dx_n \wedge d\xi_n > 0 \) if \( a_0 > 0 \). The superintegral does not depend on the orientation of \( \mathbb{R}^n \), but it does depend on a choice of scalar product on \( \mathbb{R}^n \).
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Ordinary exterior differentiation, \( d \), acts on superforms and we also define

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In particular

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\dd d^\# |x|^2/2 = \sum dx_j \wedge d\xi_j = \beta,
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the *metric form*. 
Supercurrents

A supercurrent of bidimension \((p, q)\) is a linear form on the space of compactly supported superforms with the usual topology.

\[ T_{IJ} = \sum_{|I| = n - p, |J| = n - q} T_{IJ} d\xi_j \]

where \(T_{IJ}\) are distributions (for us, mostly measures).

Notice that a 'superfunction' is a function on \(\mathbb{R}^n\).

Therefore, a 'supermeasure', i.e., an \((n, n)\)-current of order zero, is a measure on \(\mathbb{R}^n\).

For instance (following Bedford-Taylor) we can define \((dd^\# \varphi)^{n/n!}\) for \(\varphi\) convex and not necessarily smooth. It is the Alexandrov Monge-Ampère measure of \(\varphi\).
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Let $V$ be a hyperplane in $\mathbb{R}^n$. Its complexification $V_s$ is a complex hyperplane in $\mathbb{R}^n_s$ so we can (super)integrate $(n - 1, n - 1)$-forms over $V_s$.
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A short computation gives that

$$[V]_s = [V] \wedge n^\#,$$

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A subspace of codim $p$ defines a supercurrent in the same way

$$[V]_s = c_p [V] \wedge n_1^\# \wedge ... n_p^\#.$$
Submanifolds

If $M$ is a submanifold of $\mathbb{R}^n$ of dimension $m$ and codimension $p = n - m$, we define its associated supercurrent by

$$[M]_s = c_p[M] \wedge n_1^\# \wedge ... n_p^\# = (\ast dS_M) n_1 \wedge n_1^\# \wedge ... n_p \wedge n_p^\#,$$

where $n_j$ form an ON-basis for its normal space.

Is it closed?

When $p = 1$ we have

$$d[M]_s = -c_p[M] \wedge F,$$

where $F = dn_\#$.

This is (when restricted to $M$) the second fundamental form of $M$, the derivative of the Gauss map. This vanishes only when $n$ is constant, i.e. $M$ is a linear subspace. But, ...
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$$|M| = \int [M]_s \wedge \beta^m / m!.$$

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Minimal submanifolds

A small computation gives

\[ d[M]_s \wedge \beta^{n-2} / (n - 2)! = tr(F)n^\# [M]_s \wedge \beta^{n-1} / (n - 1)! \]
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Here \( tr(F) = H \) is the trace of the second fundamental form and \( H\bar{n} \) is the mean curvature vector.
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is closed if and only if the mean curvature vanishes, i.e. \( M \) is a minimal manifold.
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In general dimension \( m \)

\[ d[M]_s \wedge \beta^{m-1} / (m - 1)! = \sum \text{tr}(F_j) n_j^\# [M]_s \wedge \beta^m / m! , \]

with \( F_j = dn_j^\# \) and \( \sum \text{tr}(F_j) \vec{n}_j \) is again the mean curvature vector.
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with \( F_j = d n_j^\# \) and \( \sum tr(F_j) \vec{n}_j \) is again the mean curvature vector. So, \( [M]_s \wedge \beta^{m-1} \) is closed precisely when \( M \) is minimal.
Note that $S := [M]_s \wedge \beta^{m-1}/(m - 1)!$ is of bidegree $(n - 1, n - 1)$, i.e., of bidimension $(1, 1)$. But it is not an arbitrary $(n - 1, n - 1)$-current; it has the form $S = A \wedge \beta^{m-1}$, where $A \geq 0$. 
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Now assume that \( M \) is minimal, so that \( S \) is closed. If \( u \) is a 
function

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dd^\# u S = (dd^\# u) \wedge S.
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This defines a Laplace operator on \( M \) which has no first order 
or second order terms, just like on a complex manifold.
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This defines a Laplace operator on $M$ which has no first order or second order terms, just like on a complex manifold. One verifies that the Newton kernel

$$E_{m-2} := - (1/(m - 2)) \frac{1}{|x|^{m-2}}$$

is subharmonic on $[M]_s$. 


Volume computation à la Lelong

We look at the volume of $M$ intersected with a ball of radius $r$

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From this we get the monotonicity theorem; $\sigma(r)/r^m$ is increasing.
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From this we get the monotonicity theorem; $\sigma(r)/r^m$ is increasing. We also get that the Laplacian of $E_{m-2}$ on $M$ contains a point mass at the origin.
General domains

The proof used that $|x|$ is constant on the boundary of the ball.
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$$|M| \geq \omega_m w(a).$$
As a consequence we get a result by Alexander-Osserman and Brendle-Hung:

**Theorem**

Let $a$ be a point in the unit ball. Let $M$ be an $m$-dimensional minimal manifold in the ball that contains $a$. Then

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So we can choose $w(x) = (1 + |a|^2 - 2a \cdot x)^{m/2}$, 
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Mean curvature flow

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Recall that $\vec{H} := \sum tr(F_j)\vec{n}_j \ (F_j = dn^#)$ is the mean curvature vector field. It does not depend on the choice of ON-basis $n_j$. 
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The flow exists for short times, but always collapses in finite time. (Look at a sphere.)
Recall that

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Hence \( dd^\# S \) has the form
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By Cartan’s formula, this is the Lie derivative of \( \sigma \) along the flow (since \( d\sigma = 0 \)). Keeping track of signs etc we get
Theorem

Let $M_t$ be moving under the mean curvature flow. Then

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\frac{d}{dt} [M_t]_s \wedge \beta^m / m! = -|\vec{H}|^2 [M_t]_s \wedge \beta^m / m! - dd^\# S.
$$

Integrating this we see that the volume decreases under the mean curvature flow. Integrating against a function $\rho$ we get

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\frac{d}{dt} \int_{M_t} \rho \, dV_t = -\int_{M_t} \rho |\vec{H}|^2 \, dV_t - \int dd^\# \rho \wedge S.
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If $\rho$ is convex, this is negative, so $\int_{M_t} \rho \, dV_t$ decreases. As a consequence, if $M_0$ is contained in a convex set, $M_t$ stays there.
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Thanks!