Metric geometry of singularity types

Joint work with T. Darvas and C. Lu

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Goal of today $=$ study the *singularities* of **quasi-plurisubharmonic functions !**

Quasi-plurisubharmonic functions: definition

Fix θ a real closed $(1,1)$-form (no positivity condition)

Definition

Let $PSH(X,\theta)$ denote the set of L^1-functions $u : X \to \mathbb{R} \cup \{-\infty\}$ s.t. locally $u = smooth + psh$ and $\theta + i\partial\overline{\partial}u \geq 0$ ("weak" sense).

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Example: $\theta = \omega$=Kähler, $V_\omega = 0$.

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Goal of today: Define a (pseudo)-metric $d_{\mathcal{S}}$ on this space!

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  - If u, u_j bounded: **Bedford-Taylor theory** '82
  - If u, u_j singular: **Boucksom-Eyssidieux-Guedj-Zeriahi** '10

Monge-Ampère masses

If u, u_j are smooth or bounded

$$\int_X \theta_u^n = \int_X \theta_{u_1} \wedge \cdots \wedge \theta_{u_n} = vol(\{\theta\}) := \int_X \theta_{V_\theta}^n > 0 \quad (\leftrightarrow \{\theta\} \textit{ is big})$$

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Warning: The zero mass case is problematic.

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The reverse implication is not true:
Consider $\theta = \omega =$Kähler and $u \sim -(-\log \|z\|)^\alpha$, $\alpha \in (0,1)$, then

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$\rightsquigarrow$ Look for the *least singular* function with a given mass...

The ceiling operator $\mathcal{C}$

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FACTS:

- $\mathcal{C}(u)$ is θ-psh
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Example: Let $u \in PSH(X, \theta)$ be s.t. $\int_X \theta_u^n = vol(\{\theta\})$ then $\mathcal{C}(u) = V_\theta$.

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- If $[u], [v] \in \mathcal{S}(X, \theta)$ with $[u] \leq [v]$ then

$$d_{\mathcal{S}}([u],[v]) = \frac{1}{n+1} \sum_{k=0}^{n} \left(\int_X \theta_{V_\theta}^k \wedge \theta_v^{n-k} - \int_X \theta_{V_\theta}^k \wedge \theta_u^{n-k} \right) \geq 0$$

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- In general, there exists an absolute constant $C = C(n) > 1$:

$$d_{\mathcal{S}}([u],[v]) \leq \sum_{k=0}^{n}\left(2\int_X \theta^k_{V_\theta}\wedge\theta^{n-k}_{\max(u,v)} - \int_X \theta^k_{V_\theta}\wedge\theta^{n-k}_v - \int_X \theta^k_{V_\theta}\wedge\theta^{n-k}_u\right) \leq C d_{\mathcal{S}}([u],[v]).$$

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Theorem (DDL'19)

$(\mathcal{S}(X,\theta), d_{\mathcal{S}})$ *is a pseudo-metric space. Also,*

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Then $(\mathcal{S}_\delta(X,\theta), d_\mathcal{S})$ is complete.

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Note: Convergence results for MA measures of singular functions are not trivial at all!

Applications: Semicontinuity of multiplier ideal sheaves

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"Version" of the **strong openess theorem** *conjectured by Demailly '00 and proved by Guan-Zhou '15,'16.*

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Jump into the past (DDL '17,'18):

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Theorem (DDL'18)

Let $\phi = \mathcal{C}(\phi)$ be s.t. $\int_X \theta_\phi^n > 0$. Assume

$$\int_X f\omega^n = \int_X \theta_\phi^n.$$

Then there exists a unique ψ ($\sup_X \psi = 0$) solution of (MA_ϕ)

"Historical" Note: it is a generalisation of the **Calabi-Yau theorem**

Back to the present: Work with a family of Monge-Ampère equations

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Theorem (DDL'19)

Solutions to a family of Monge-Ampère equations with varying singularity type converge as governed by the $d_\mathcal{S}$-topology.
More precisely, $\|\psi - \psi_j\|_{L^1} \to 0$.