Oscillation theory of plurisubharmonic functions and Bergman kernel estimates

– Joint work with Bo-Yong Chen

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A nice application of the Chebyshev polynomial is the following

Theorem (Remez inequality)

Let p be a degree d polynomial on $\mathbb{R}$. Then

$$\sup_I |p| \leq c_d \cdot \sup_E |p|,$$

I: interval, $E \subset I$: measurable. The sharp constant

$$c_d = T_d\left(2 \cdot \frac{|I|}{|E|} - 1\right) \leq \left(4 \cdot \frac{|I|}{|E|}\right)^d,$$

where T_d is the Chebyshev polynomial

$$T_d = \begin{cases} \cos(d \cos^{-1} x) & |x| \leq 1, \\ \frac{1}{2}\left((x + \sqrt{x^2 - 1})^d + (x - \sqrt{x^2 - 1})^d\right) & |x| > 1. \end{cases}$$

Associated upper oscillation estimate

Let us assume that $\sup_I |p| = 1$. Take

$$E_t = \{|p| \leq e^{-t}\}, \quad t \geq 0.$$

Apply the Remez inequality to E_t:

$$1 \leq \left(\frac{4|I|}{|E_t|}\right)^d e^{-t} \Leftrightarrow |E_t| \leq 4e^{-t/d}|I|,$$

which gives

$$-\int_I \log|p| = \int_0^\infty |E_t|\, dt \leq \int_0^\infty 4e^{-t/d}|I| = 4d|I|.$$

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Observation (upper oscillation estimate)

$$\sup_I \log|p| - (\log|p|)_I \le 4\deg p, \qquad (\log|p|)_I := \frac{1}{|I|}\int_I \log|p|.$$

Different kinds of oscillations

A-Oscillation of u:

$$AO_I u := \frac{1}{|I|}\int_I |u - A|.$$

When A is the average, we get the well known *mean oscillation*

$$MO_I u := \frac{1}{|I|}\int_I |u - u_I|.$$

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Observation (A simple observation)

$$MO \leq 2UO.$$

First main result

Remez inequality gives

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Theorem (Upper oscillation for polynomials)

For all non-empty compact convex set $A \subset \mathbb{C}^n$, we have

$$UO_A \log|p| \leq \gamma \cdot \deg p, \quad \forall\ p \in \mathbb{C}[x_1, \cdots, x_n].$$

Here the sharp constant $1.278 < \gamma < 1.279$ is determined by

$$\gamma + \log(\gamma - 1) = 0.$$

It comes from the line segment type UO bound for $\log|z|$, i.e.

$$\gamma := \sup_{a,b\in\mathbb{C}} UO_{[a,b]} \log|z|, \quad [a,b] : \textit{ line segment between } a, b.$$

About the proof

First, assume that $p \in \mathbb{C}[z]$, then

$$p = a_0(z - a_1)^{n_1} \cdots (z - a_k)^{n_k},$$

thus

$$\sup_{[a,b]} \log |p| \leq \log |a_0| + \sum_{j=1}^{k} n_j \sup_{[a,b]} \log |z - a_j|$$

and

$$(\log |p|)_{[a,b]} = \log |a_0| + \sum_{j=1}^{k} n_j (\log |z - a_j|)_{[a,b]}.$$

Thus

$$UO_{[a,b]}(\log |p|) \leq \sum_{j=1}^{k} n_j \gamma = \gamma \cdot \deg p.$$

About the proof

For general $p \in \mathbb{C}[z_1, \cdots, z_n]$, since A is compact, we may choose $z_0 \in A$ such that

$$|p(z_0)| = \sup_{z \in A} |p(z)|.$$

For every ray (half line), say L, starting from z_0, we see that $A \cap L$ is a line segment in view of *convexity* of A. Let $L_{\mathbb{C}}$ be the complex line containing L. Apply our theorem to $p|_{L_{\mathbb{C}}}$, we have

$$UO_{A\cap L}(\log |p|) = UO_{A\cap L}(\log |p|_{L_{\mathbb{C}}}|) \leq \gamma \deg p|_{L_{\mathbb{C}}} \leq \gamma \deg p,$$

which gives

$$UO_A(\log |p|) \leq \gamma \deg p$$

since $UO_A(\log |p|)$ is a certain average of $UO_{A\cap L}(\log |p|)$ for all L starting from z_0 by the choice of z_0.

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Corollary (weak Remez type inequality)

For every non-empty compact convex set A in $\mathbb{C}^n$ we have

$$\sup_A |P| \leq \sup_E |P| \cdot \left(e^{2\gamma} \frac{4n|A|}{|E|}\right)^{\deg P}, \quad e^{2\gamma} \leq e^3,$$

for every $P \in \mathbb{C}[z_1, \cdots, z_n]$ and measurable $E \subset A$.

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Remark (Sharp version)

A relatively sharp version of the above estimate was obtained by Yu. Brudnyi and Ganzburg in 1973.

A result of Siciak (can be proved using Hörmander L^2 estimate) is:

— PSH functions in the Lelong class ($\leq \log(|z|+1)$) can be approximated by $(\log|p|)/d$, $d \geq \deg p$.

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Theorem (Second main theorem)

$PSH \subset BUO(polydisc_N) \subset BUO(ball)$, $PSH \not\subset BMO(polydisc)$.

$P \in polydisc_N \Leftrightarrow \max\{r_j\} \leq \min\{r_j^{1/N}\}$.

Corollary of our main theorem

Together with the John-Nirenberg inequality, our second main theorem gives the following:

Corollary

Assume that $\phi \in \mathrm{psh}(\Omega)$. *Fix* $\Omega_0 \Subset \Omega$. *Then for every* $N > 0$, *there exists* $\varepsilon > 0$ *such that*

$$\frac{1}{|P|}\int_P e^{-\varepsilon(\phi - \sup_P \phi)} \leq C(N, \varepsilon),$$

for all type-N polydiscs P *in* Ω_0.

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Remark (Sharp ε ?)

We do not know how to find the sharp ε.

Associated Bergman kernel estimate

Recall that the Bergmen kernel is defined by

$$K_{\phi,P}(z) := \sup_{f\in\mathcal{O}(P)} \frac{|f(z)|^2}{\int_P |f|^2 e^{-\phi}},$$

Take $f \equiv 1$, we get

$$K_{\phi,P}(z) \geq \frac{1}{\int_P e^{-\phi}}.$$

On the other hand, mean value identity for $f \in \mathcal{O}(P)$ gives

$$|f(0)|^2 = |\frac{1}{|P|}\int_P f|^2 \leq \frac{1}{|P|}\int_P |f|^2 e^{-\phi} \cdot \frac{1}{|P|}\int_P e^{\phi},$$

where 0 denotes the center of P. Thus

$$\left(\frac{1}{|P|}\int_P e^{-(\phi-\sup_P \phi)}\right)^{-1} \leq K_{\phi,P}(0)\cdot|P|\cdot e^{-\sup_P \phi} \leq 1$$

John Nirenberg inequality implies: $K_{\varepsilon\phi,P}(0)\cdot|P| \sim e^{\sup_P \varepsilon\phi}$ for small ε. Take $P = P_{r^a} := \mathbb{D}_{r^{a_1}} \times \cdots \mathbb{D}_{r^{a_n}}$.

Directional Lelong number

We get

$$\lim_{r\to 0+} \frac{\log\left(K_{\varepsilon\phi,P_{r^a}}(0)\cdot|P_{r^a}|\right)}{\varepsilon\log r} = \lim_{r\to 0+} \frac{\sup_{P_{r^a}}\phi}{\log r}.$$

The right hand is precisely the *Lelong number* along direction $a=(a_1,\cdots,a_n)$.

The left hand side is also interesting since

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Theorem (Berndtsson's theorem)

$\log K_{\phi, P_{r^a}}(0)$ *is a convex function of* $\log r$.

Three one dimensional examples

Example ($\phi(0) > -\infty$)

$r^2 K_{\phi,\mathbb{D}_r}(0)$ *is log-convex increasing wrt* $\log r$. *Thus*

$$K_{\phi,\mathbb{D}}(0) \geq \lim_{r\to 0} r^2 K_{\phi,\mathbb{D}_r}(0) = \frac{e^{\phi(0)}}{\pi}.$$

Example ($\lim_{|z|\to 0} \phi(z) - \varepsilon \log|z| = A, \quad 0 \leq \varepsilon < 2$)

$r^{2-\varepsilon} K_{\phi,\mathbb{D}_r}(0)$ *is log-convex increasing wrt* $\log r$. *Thus*

$$K_{\phi,\mathbb{D}}(0) \geq \lim_{r\to 0} r^{2-\varepsilon} K_{\phi,\mathbb{D}_r}(0) = \frac{e^{A}}{2\pi(2-\varepsilon)}.$$

Example ($\lim_{|z|\to 0} \phi(z) - 2\log|z| - 2\log(-\log|z|) = B$)

$K_{\phi,\mathbb{D}_r}(0)/|\log r|$ *is log-convex increasing wrt* $\log r$.

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Remark

One also gets the equivalent Ohsawa–Takegoshi extension type theorems wrt

$$\int |f|^2 e^{-\phi}, \quad \int \frac{|f|^2}{|z|^\varepsilon} e^{-\phi}, \quad \int \frac{|f|^2}{|z|^2(\log|z|)^2} e^{-\phi}$$

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Remark (When ϕ is smooth and sh)

$$\lim_{r\to 0} \frac{1}{r} \frac{d \log r^2 K_{\phi,\mathbb{D}_r}(0)}{dr} = \phi_{z\bar{z}}(0).$$

A question when ϕ is smooth

Assume that ϕ is smooth and

$$\omega := i\partial\overline{\partial}\phi > 0$$

on $\mathbb{D}$. It is known that

$$\lim_{m\to\infty} \frac{K_{m\phi,\mathbb{D}}(0)e^{-m\phi(0)}}{m} = \frac{\phi_{z\bar{z}}(0)}{\pi}$$

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Question (May not be correct!)

Assume that $|Ric(\omega)| < 1$ and $\int_{\mathbb{D}} \omega < 1$. Then there exist absolute constants ε and N such that

$$\frac{K_{m\phi,\mathbb{D}}(0)e^{-m\phi(0)}}{m} \geq \varepsilon \cdot \frac{\phi_{z\bar{z}}(0)}{\pi}$$

for all $m \geq N$.