

Specialization for the pro-finite fundamental group

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K NA field $\theta = \theta_K$ $k = \theta/\mathfrak{m}$

Thm (Grothendieck, SGA 1).

X/θ proper flat scheme, connected fibres

(a) $\exists$ natural specialization map

$$\text{sp: } \pi_1^{\text{et}}(X_K) \longrightarrow \pi_1^{\text{et}}(X_k)$$

(b) it is surjective if X is normal or X_k is reduced

Ex (Tate curve). $K = \bar{K}$ char 0

$$\begin{array}{ccc} X_K \text{ (circle)} & \xrightarrow{\pi_1^{\text{et}}(X_K)} & \pi_1^{\text{et}}(X_k) \text{ (circle)} \\ \downarrow \cong & & \downarrow \cong \\ \hat{\mathbb{Z}} & \xrightarrow{\pi} & \hat{\mathbb{Z}} \end{array}$$

Note. X_k has a $\mathbb{Z}$ -covering by $\dots$  $\dots$ X_K does not (it is normal)

Thm (ALY '21)

$X/\mathbb{Q}$ formal scheme, l.f.t. (don't need paper!)

$$(a) \exists \begin{array}{ccc} \pi_1^{dJ}(X_{rig}) & \xrightarrow{sp} & \pi_1^{dJ}(X_k) \\ \downarrow & & \downarrow \\ \pi_1^{dJ}(X_{rig}) & \xrightarrow{(SGA)} & \pi_1^{dJ}(X_k) \end{array}$$

(b) sp has dense image if X is η -normal

$$\begin{array}{ccc} \hat{Z} \times \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \hat{Z} \times \hat{\mathbb{Z}} & \longrightarrow & \hat{\mathbb{Z}} \end{array}$$

§ 1. $\exists_1^{p,q}$ (Bhatt-Johalve)

X loc. log. Noeth scheme

Def. a geometric covering is an étale $Y \rightarrow X$ satisfying VCP

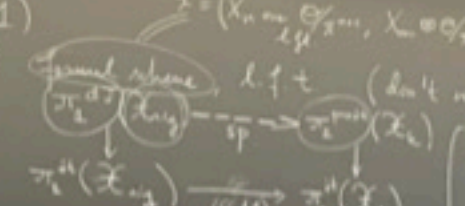
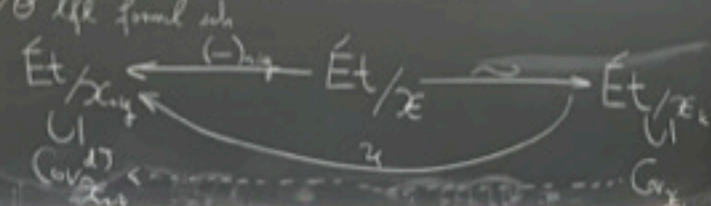
$$\text{Cov}_X = \{ \text{geom. covs of } X \}$$

Ex 1) X normal $\Rightarrow Y = \text{chain of } \mathbb{P}^1$'s

2) X normal $\Rightarrow \text{Cov}_X = \{ \text{disj. unions of fin. étale} \}$

3) $X = G_m \vee G_m \Rightarrow \text{Cov}_X$ is complicated

$\text{Cov}_X^{d\tau}$
 $Y \rightarrow X \rightarrow Z$
 $\perp \perp$ fin. state.
 of paper opens
 pt
 a same infinite Galois at
 (F_{∞}) .

Then (ALY '21)
 $X/\mathcal{O}$
 (a) $\exists$  (don't need paper!)
 (b) sp has dense image if X is η -normal
 § 3. The specialisation map $\pi_s^{d\tau} \rightarrow \pi_s^{p+d}$
 $X/\mathcal{O}$ lft formal sch


So if X_i are normal then we're done!
 • raises hope: $\exists$ also blowup $X' \rightarrow X$ s.t.
 the inv. comps of X'_i are normal.
 • instead: Prop $\exists$ also blowup $X' \rightarrow X$ s.t. $\forall$ inv. comp Z_i
 with image $Z \subset X_i$ the map $Z' \rightarrow Z$ factors
 through the normalisation of Z .
 Proof sketch: idea $|X_i| = Y_i \cup \dots \cup Y_r$ inv. comp
 Y_i = completion of X along $\mathcal{O}_i$
 $(Y_i)_{d\tau} = \text{"dub" of } Y$
 $\Rightarrow X_{\text{reg}} = \bigcup (Y_i)_{d\tau}$ is open!

