

A logarithmic HKR theorem and residue sequences for (log) HH

(with T. Lundemo, D. Park, P.A. Østvær).

§1. Introduction.

$R \in \text{CRing}$, $A \in \text{Algr}_R$.

Then, Thm(HKR): $\nexists A/R$ smooth. $\Omega_{A/R}^* \xrightarrow{\sim} \text{HH}_*(A/R)$
is of graded rings.

$$\text{HH}(A/R) = A \bigotimes_{A \bigotimes_R A}^L A$$

Following Quillen, the proof goes in 2 steps.

(A) Show HKR holds for free algebras.

(B) Deduce the gen. statement after using AQ spectral sequence.

$$E_{p,q}^2 = \pi_p(\Lambda^q L_{A/R}) \Rightarrow HH_{pq}(A/R)$$

$\nexists A/R$ smooth. $L_{A/R} = \Omega_{A/R}[1] \leadsto$ 2 seq. columns.

We like HH (and variants) because of its relat. with K-theory

But unlike K-theory, HH does not satisfy localization.

(Rognes, Hesselholt, Madsen) you need to consider log structures

Ex $\nexists A$ is DVR, then there is a cofiber seq. $K = A/\pi$

$$HH(A) \rightarrow \boxed{\log HH(A, A-d\log)} \rightarrow HH(K)[1]$$

$$K(A) \rightarrow K(F) \rightarrow \Sigma K(K)$$

§2. log rings and log HH

Def: A (discrete) pre-log ring is the datum of $\alpha: M \rightarrow A$, $A \in \mathbf{CRing}$, $M \in \mathbf{dMon}$
 A log ring is a pre-log ring such that $\alpha^{-1}(A^\times) \cong A^\times$.
 d not between M and $(A, \cdot)$

Ex:
 (0) $A \in \mathbf{CRing}$, $(A, A^\times)$ trivial log structure on A
 (1) A domain
 $K = \text{Frac}(A)$ $(A, A^\times) \rightarrow (A, A^\times) \rightarrow (K, K^\times)$

(2) A DVR, π uniformiser of A . $N \rightarrow A \xrightarrow{\pi} A/\pi = k$ standard log point on k

(3) $N^r \rightarrow R[t_1, \dots, t_m]$, $e_i \mapsto t_i$

Def: Derived pre-log ring / animated pre-log ring.

Can be modelled by $\mathbf{sCRing} \ni A$ $M \rightarrow A$

Def: $M \in \mathbf{dMon}$, $\mathbf{sMon} \ni M$
 $\Omega BM = M^{\text{gp}}$

given a map $N \rightarrow M$ with a retraction $N \rightarrow M \rightarrow N$

$$\mathbb{R}: H^{\text{top}} = H^{\text{top}}_N = N \times H^{\text{top}}$$

Relative relation / combination of H/N

$$\mathbb{R} \text{ (Rogers)} \quad (R, D) \rightarrow (A, M) \quad \text{+ dP}_{\text{rel}}$$

$$B^{\text{rel}}_P(M) = S^1 \oplus_P M \quad \text{augmented / M}$$

$$B^{\text{rel}}_P(M) = [S^1 \oplus_P M - M]^{\text{rel}}$$

$$\log HH((A, M)/R, P) = HH(A/R) \oplus \frac{Z(B^{\text{rel}}_P(M))}{Z(B^{\text{rel}}_P(M))}$$

But unlike K-theory, HH does not satisfy localization
 (Rognes, Hesselholt, Madsen) you need to consider log structures

Thm (BLPØ): $\log HH(A, M)/(R, P)$ admits a descending sp. filtration (top line pres.)
 with graded pieces given by $\Lambda^i \mathbb{L}_{(A, M)/(R, P)}[i]$

In fact, there is a conv. sp. seq.

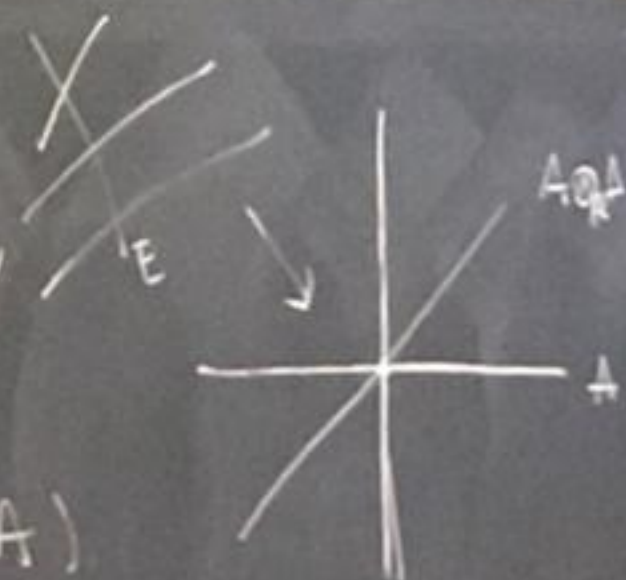
$E_{p, q}^2 = H_p(\Lambda^q \mathbb{L}) \Rightarrow \pi_{p, q}(\log HH(A, M)/(R, P))$
 $\mathbb{L}_{(A, M)/(R, P)}$ is derived log smooth, with $(A, M)/(R, P)$ cocomplete
 then, the sp. seq. collapses $\Omega_{(A, M)/(R, P)}^* \rightarrow \log HH(A, M)/(R, P)$.

$$M \in \text{dMon}, \quad \text{SCMon} \ni M \quad M \rightarrow A$$

$$\Omega BM = M^{\text{op}}$$

Differentials

$$(A, M)/(R, P) \quad \Omega' A/R \xrightarrow{\langle da \rangle} \Omega' (A, M)/(R, P) \xrightarrow{\langle da, \frac{dm}{m} \rangle} \Omega' A[M^{-1}] \xrightarrow{\langle \frac{da}{m} \rangle}$$



Recall. $\Omega' A/R \cong I/I^2$, $I = \text{Ker}(A \otimes_R A \rightarrow A)$

Try to do the same in log case?

look at the repletion of the diagonal.

$$I = \text{Ker}\left((A \otimes_R A) \otimes_{\mathbb{Z}[M \oplus_P M]} \mathbb{Z}[M \oplus_P M]^{\text{wp}} \xrightarrow{F} A \right), \quad I/I^2 = \Omega' (A, M)/(R, P)$$

Extend to derived rings $\leadsto$ by taking the log canonical resolution
 get a normal orb of compact type $\mathbb{L}(A, M)/(\mathbb{R}, P)$

Prop. X/S derived log scheme

$L_{X/S}$ = derived cotangent bundle
 of the log scheme

$$X \xrightarrow{\Delta} X \times X$$

$$\swarrow \quad \uparrow$$

$$X \times X \xrightarrow{\Delta} X \times X$$

(cf. SSV) define derived log orb maps $\mathbb{L}(A, M)/(\mathbb{R}, P) \geq 1$

$\Rightarrow$ on π derived log orb

Way around: integral log orb maps of derived log rings $\Rightarrow L \otimes 1$

Similarly, $\mathbb{Q}(A, M)/(\mathbb{R}, P) \neq \mathbb{L}(A, M)/(\mathbb{R}, P)$ for log smooth, is equal, or for integral

General way to kill this discrepancy is to consider the orbifold structure.

Can be modeled by $\delta \text{Char}_X \ni A$

Thm $\log \text{HH}(A, M_Y(R, P)) \cong A \otimes_{A \otimes A} A$

Main Theorem follows by (1) direct comp in the free case
(2) Quillen's proof

Motives : S is log scheme of finite Krull dimension

$$\log \text{DA}^{\text{eff}}(S) = \bigvee_{\alpha \in \text{Nis}} (\square_{\alpha}) \left(\text{Spa}(\text{PSH}(\mathcal{L}_{\text{Sm}}/S)) \right)$$

$\downarrow$
 $\log \text{DA}(S)$

$$\overline{D} = (\mathbb{P}^1, \infty)$$

(Rognen, Hesselholt, Madsen) : you need to consider log structures

Thm : $\log HH_{dNis}(-/S)$ is representable

forget about this if S has volutative log struct.

Cor : if X/S smooth, $Z \hookrightarrow X$ smooth subscheme.

If S has no log str.

$\Rightarrow$

$$HH(Z/S) \rightarrow HH(X/S) \rightarrow \log HH(B_{\mathbb{Z}}^{\wedge}(X), E)$$

$$\begin{array}{c} K(X) \\ \downarrow \\ K(B_{\mathbb{Z}}^{\wedge}(X) - E) \end{array}$$

$$K(X)$$