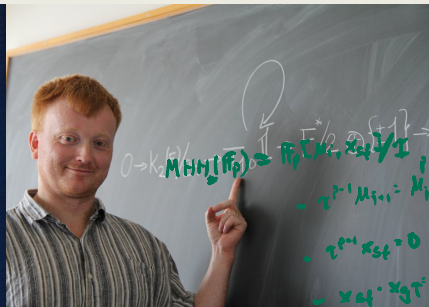


Motivic Hochschild homology of prime fields

Bjørn Ian Dundas
j/w Mike Hill, Kyle Ormsby, Paul Arne Østvær

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Meditating

Handling and

Hey!!

Motivic Hochschild homology, j/w Hill, Ormsby, Østvær

Why?

- ▶ Import important tool to motivic world
 - ▶ reveals arithmetic structure
 - ▶ starting point for all trace results for algebraic K-theory
- ▶ Shed light on topological case

What?

$$HH^S(C) = C \otimes_{C^{\mathrm{op}} \otimes_S C} C,$$

where C is an S -algebra.

Today:

- ▶ S is the motivic sphere spectrum over an algebraically closed field of char $\neq p$ and
- ▶ $C = M\mathbb{F}_p$ is the mod p motivic Eilenberg-Mac Lane spectrum

How? Muddling through as usual, with the complications and benefits of the motivic world.

Abstract: $HH_*(\mathbb{F}_p) = \dots$ in the motivic world over an algebraically closed field of characteristic different from p .

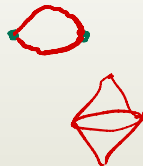
What *is* “HH”? (bear with me) – it’s the circle in a context

What is the circle,
aka Hochschild homology,
in a given context?

- ▶ Algebra: $S = \mathbb{Z}$, the integers:
“Shukla homology” of C
- ▶ Topology: S the sphere spectrum:
“topological Hochschild homology” of C
- ▶ Motivic: S the motivic sphere spectrum:
“motivic Hochschild homology” of C .

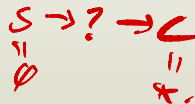
If $\mathcal{C}$ is a model category with an
initial object $\emptyset$ and final object $*$
we **define the spheres**^a

- ▶ $S^0 = * \coprod_{\emptyset} *$
- ▶ $S^1 = * \coprod_{S^0} *$
- ▶ etc. . . $S^{n+1} = * \coprod_{S^n} *$



If $\mathcal{C} =$ commutative S -algebras over an
 S -algebra C :^b

- ▶ $S_C^0 = C \otimes_S C$,^c
- ▶ $S_C^1 = C \otimes_{C \otimes_S C} C$.^d



^aall constructions derived

^bso that S is initial and C is final

^cthe “dual Steenrod algebra” (of C)

^d“Hochschild homology” of C

What *is* the dual Steenrod algebra $S_{\mathbb{F}_p}^0 = \mathbb{F}_p \otimes_S \mathbb{F}_p$?

- Algebra: $S = \mathbb{Z}$, the integers, $C = \mathbb{F}_p$:

$$\pi_* S_{\mathbb{F}_p}^0 = \mathbb{F}_p[\tau_0]/\tau_0^2, \quad |\tau_0| = 1.$$

- Topology: S the sphere spectrum, $C = H\mathbb{F}_p$, the mod p EM-spectrum:

$$\pi_* S_{H\mathbb{F}_p}^0 = \begin{cases} \mathbb{F}_2[\tau_0, \tau_1, \dots] & p = 2 \\ \mathbb{F}_p[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/\tau_i^2 & p \text{ odd} \end{cases}$$

- Motivic: S the motivic sphere spectrum over a field of char. $\neq p$,
 $C = M\mathbb{F}_p$, the mod p motivic EM-spectrum:

$$\pi_* S_{M\mathbb{F}_p}^0 = \begin{cases} \pi_* M\mathbb{F}_2[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/\tau_i^2 + \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1} & p = 2 \\ \pi_* M\mathbb{F}_p[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/\tau_i^2 & p \text{ odd} \end{cases}$$

with $|\tau| = (0, -1)$, $|\rho| = (-1, -1)$, $|\tau_i| = (2 \cdot p^i - 1, p^i - 1)$, $|\xi_i| = (2 \cdot p^i - 2, p^i - 1)$.

This column just stores stuff you may have forgotten: everything happens to the left

► $S^0 = * \coprod_{\emptyset} *$

► $S^1 = * \coprod_{S^0} *$

C = commutative S -algebras over an S -algebra C :

► S_C^0 is "the dual Steenrod algebra" $C \otimes_S C$,

► S_C^1 is "Hochschild homology" $HH^S(C) = C \otimes_{C \otimes_S C} C$.

$$\pi_* H\mathbb{F}_p = \mathbb{F}_p$$

What *is* (... Hochschild homology) $S_{\mathbb{F}_p}^1 = \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes_S \mathbb{F}_p} \mathbb{F}_p$?

- Algebra: S the integers:

$$\pi_* S_{\mathbb{F}_p}^1 = \Gamma(\mu_0) \quad \boxed{\text{divided powers}} \quad |\mu_0| = 2.$$

- Topology: S the sphere spectrum:

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\mu_0] \quad \boxed{\text{polynomial,}} \quad |\mu_0| = 2.$$

- Motivic: S the motivic sphere spectrum (k alg. closed):

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\tau][\mu_i, x_{S,f}]/\mathcal{I}$$

- $\tau^{p-1} \mu_{i+1} = \mu_i^p$ " τ^{p-1} -divided powers"
- $\tau^{p-1} x_{S,f} = 0$ " τ^{p-1} -torsion"
- $x_{S,f} \cdot x_{T,g} = \sum_u \epsilon_u \cdot x_{S \cup T \cup \{u\}, f+g}$ $\epsilon_u \in \mathbb{F}_p$, indexing tbe.

$\mathcal{C}$ = commutative S -algebras over an S -algebra C :

"dual Steenrod algebra"

$$S_C^0 = C \otimes_S C, \text{ e.g.,}$$

- Algebra: $S = \mathbb{Z}$,
 $\pi_* S_{\mathbb{F}_p}^0 = E(\tau_0)$

- Topology: $S = \mathbb{S}$,
 $\pi_* S_{\mathbb{F}_p}^0 =$

$$\begin{cases} \bigotimes_{i \geq 0} \mathbb{F}_2[\tau_i] \\ \bigotimes_{i \geq 0} E(\tau_i) \otimes \mathbb{F}_p[\xi_{i+1}] \end{cases}$$

"Hochschild homology"

$$S_C^1 = C \otimes_{C \otimes_S C} C.$$

Motivic HH (S the motivic sphere spectrum/alg closed field of char $\neq p$)

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\tau][\mu_i, x_{S,f}]/\mathcal{I}$$

Relations

- ▶ $\tau^{p-1} \mu_{i+1} = \mu_i^p$ " τ^{p-1} -divided powers"
- ▶ $\tau^{p-1} x_{S,f} = 0$ " τ^{p-1} -torsion"
- ▶ $x_{S,f} \cdot x_{T,g} = \sum_{u \in \text{supp}(f+g) - S \cup T} \epsilon_u \cdot x_{S \cup T \cup \{u\}, f+g}$

but does
have poly
no. or

Indexing:

- ▶ $f: \mathbb{N} \rightarrow \mathbb{N}$ finite non-empty support,
- ▶ $S \subset \text{supp} f$ not containing min $\text{supp} f$

Coefficient $\epsilon_u \in \mathbb{F}_p$ given by explicit formula

Bidegrees

- ▶ $|\tau| = (0, -1)$
- ▶ $|\mu_i| = (2p^i, p^i - 1)$
- ▶ $|x_{S,f}| = (|S| + 1)(-1, p - 1) + p \sum_{j \in \text{supp} f} f(j) |\mu_j|$
so, first torsion class has degree $|x_{\emptyset, \delta_0}| = (2p - 1, p - 1)$

$p = 2$
 $(3, 1)$
 $(7, 1)$
 $(7, 3)$
 $\vdots$

Algebra: S the integers:

$$\pi_* S_{\mathbb{F}_p}^1 = \Gamma(\mu_0), \quad |\mu_0| = 2.$$

divided powers

Topology: S the sphere spt:

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\mu_0], \quad |\mu_0| = 2.$$

polynomial

Motivic world: built out of usual spaces and smooth k -varieties with the affine line $\mathbb{A}^1$ contractible and the projective line $\mathbb{P}^1$ invertible.

$$\mathbb{A}^1 = \text{Spec}(k[t])$$

$$\mathbb{G}_m = \text{Spec}(k[t, t^{-1}])$$

$$\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$$

$$\pi_{i,j} X = [S^{i-j} \wedge \mathbb{G}_m^{\wedge j}, X]$$

Motivic HH,

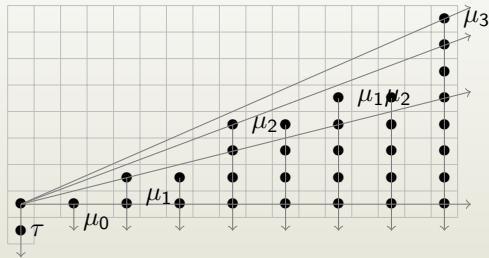
$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\tau][\mu_i, x_{S,f}]/\mathcal{I},$$

over alg closed field of char $\neq p$

τ_i ξ_i

$S=0.$

$p=2$



- ▶ $\tau^{p-1} \mu_{i+1} = \mu_i^p$ coming from the τ_i s
The motivic version of Bökstedt periodicity (with added τ -divisibility).
- ▶ $\tau^{p-1} x_{S,f} = 0$ A cloud of τ^{p-1} -torsion coming from the μ_i s inability to kill off the ξ_i s due to weight problems (as in the topological case $\tau^{p-1} \xi_i$ is killed for free)
- ▶ $x_{S,f} \cdot x_{T,g} = \sum_{u \in \text{supp}(f+g) - S \cup T} \epsilon_u \cdot x_{S \cup T \cup \{u\}, f+g}.$



Motivic HH, $\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\tau][\mu_i, x_{S,f}]/\mathcal{I}$, over alg closed field of char $\neq p$

An immediate consequence of the $x_{S,f}$ s:

- ▶ Topological case: as an $\mathbb{F}_p$ -module $\mathrm{THH}(\mathbb{F}_p) \simeq \sum_i \mathbb{F}_p \wedge S^{2i}$ because
 1. Mahowald: the mod 2 Eilenberg-MacLane spectrum is a Thom spectrum by an E_2 -map to $\Omega^2 S^3$
 2. Blumberg, Cohen, Schlichtkrull (assumptions): $\mathrm{THH}(\mathrm{Thom}) \simeq \mathrm{Thom} \wedge B(\mathrm{Base})_+$
 3. $\mathbb{F}_p \wedge B\Omega^2 S_+^3$ splits
- ▶ The τ^{p-1} torsion shows that motivic Hochschild homology is **not** a free module.
- ▶ Consequence: at least one of the above fails motivically. *← which one(s)?*

Strategy

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\tau][\mu_i, x_{S,f}]/\mathcal{I},$$

over alg closed field of char $\neq p$

1. Topology: Invert τ
2. Algebra: Mod out by τ^{p-1}
3. Intricate: Assemble in an “arithmetic square”

*magically
no
complications!*

1 and 2 can both be done via the BarSS from the Motivic Steenrod algebra

$$\pi_* S_{\mathbb{F}_p}^0 = \begin{cases} \mathbb{F}_2[\tau][\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/\tau_i^2 - \tau\xi_{i+1} & p = 2 \\ \mathbb{F}_p[\tau][\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/\tau_i^2 & p \text{ odd} \end{cases}$$

and since $p = 2$ seems most exotic we'll look at that today.

...but first...

What is $S_{\mathbb{F}_p}^1 = \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes_S \mathbb{F}_p} \mathbb{F}_p$? **Warmup** with classical cases:

Both SS

$$E^2 = \mathrm{Tor}^{\pi_* S_C^0}(\pi_* C, \pi_* C) \Rightarrow \pi_* S_C^1$$

- Algebra: $S = \mathbb{Z}$, the integers, $C = \mathbb{F}_p$:

$$E^2 = \mathrm{Tor}^{E(\tau_0)}(\mathbb{F}_p, \mathbb{F}_p) = \Gamma(\mu_0)$$

no further differentials or extensions.

- Topology: $S = \mathbb{S}$, the sphere spectrum, $C = \mathbb{F}_p$:

$$E^2 = \begin{cases} \mathrm{Tor}^{\bigotimes_{i \geq 0} \mathbb{F}_2[\tau_i]}(\mathbb{F}_2, \mathbb{F}_2) = \bigotimes_{i \geq 0} E(\mu_i) \\ \mathrm{Tor}^{\bigotimes_{i \geq 0} E(\tau_i) \otimes \mathbb{F}_p[\xi_{i+1}]}(\mathbb{F}_p, \mathbb{F}_p) = \bigotimes_{i \geq 0} \Gamma(\mu_i) \otimes E(\lambda_{i+1}) \end{cases}$$

Bökstedt: differentials and extensions **(!)**

$\mathcal{C}$ = commutative S -algebras over an S -algebra C :

“dual Steenrod algebra”

$$S_C^0 = C \otimes_S C,$$

“Hochschild homology”

$$S_C^1 = C \otimes_{C \otimes_S C} C. \text{ e.g.,}$$

- Algebra: $S = \mathbb{Z}$,
 $\pi_* S_{\mathbb{F}_p}^0 = E(\tau_0),$

$$\pi_* S_{\mathbb{F}_p}^1 = \Gamma(\mu_0)$$

- Topology: $S = \mathbb{S}$,

$$\pi_* S_{\mathbb{F}_p}^0 = \begin{cases} \bigotimes_{i \geq 0} \mathbb{F}_2[\tau_i] \\ \bigotimes_{i \geq 0} E(\tau_i) \otimes \mathbb{F}_p[\xi_{i+1}] \end{cases}$$

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\mu_0]$$

Differentials and extensions *NB!*

simpler than the original

A way to see Bökstedt's differentials and extensions that works in the motivic setting (even over non-*alg.* closed fields).

The suspension map

$$\sigma: S_{\mathbb{F}_p}^0[1] \rightarrow S_{\mathbb{F}_p}^1.$$

τ_i $\sigma \tau_i = \mu_i$
 ξ_i $\sigma \xi_i$

is not multiplicative, but power operations gives for $i \geq 0$

$$\begin{aligned} \tau^{p-1} \cdot \sigma(\tau_{i+1}) &= (\sigma \tau_i)^p, \\ \tau^{p-1} \cdot \sigma(\xi_{i+1}) &= \rho \cdot \sigma(\tau_{i+1}) \end{aligned}$$

$$\pi_* S_{\mathbb{F}_p}^0 =$$

$$\begin{cases} \pi_* M\mathbb{F}_2[\tau_i, \xi_{i+1}] / \tau_i^2 - \tau \xi_{i+1} - \rho \tau_{i+1} - \rho \tau_0 \xi_{i+1} & p = 2 \\ \pi_* M\mathbb{F}_p[\tau_i, \xi_{i+1}] / \tau_i^2 & p \text{ odd} \end{cases}$$

$$\tau \in \pi_{0,-1} M\mathbb{F}_p \cong \mu_2(F) \text{ repn by } -1$$

$$\rho \in \pi_{-1,-1} M\mathbb{F}_2 \cong F^\times / (F^\times)^2 \quad (\rho = 0 \text{ if } p \text{ odd})$$

$$p \text{ odd} \Rightarrow \rho = 0$$

$H\mathbb{F}_p \rightarrow H\mathbb{F}_p \rightarrow H\mathbb{F}_p$
 $\downarrow \quad \downarrow \quad \downarrow$
 $H\mathbb{F}_p \rightarrow H\mathbb{F}_p \rightarrow H\mathbb{F}_p$
in modules vs. in com. alg.

The 1st formula is the extension $\tau^{p-1} \mu_{i+1} = \mu_i^p$ from which – in the motivic world – all topologically hard differentials emanate from an elementary weight watch.

The 2nd is the beginning of the relations for the $x_{S,f}$ s. Also deducible from the BarSS.

The motivic dual Steenrod algebra k alg. closed field, $\text{char} \neq p$

$$\pi_*(M_{\mathbb{F}_p}) = \mathbb{F}_p[\tau], \quad |\tau| = (0, -1)$$

$$\pi_{\star} S_{\mathbb{F}_p}^0 = \begin{cases} \mathbb{F}_2[\tau][\tau_0, \tau_1, \dots, \xi_1, \dots] / \tau_i^2 - \tau \xi_{i+1} & p = 2 \\ \mathbb{F}_p[\tau][\tau_0, \tau_1, \dots, \xi_1, \dots] / \tau_i^2 & p \text{ odd} \end{cases}$$

with $|\tau_i| = (2 \cdot p^i - 1, p^i - 1)$, $|\xi_i| = (2 \cdot p^i - 2, p^i - 1)$.

For comparison:

► Algebra: $\pi_* S_{\mathbb{F}_p}^0 = E(\tau_0)$,
 $\pi_* S_{\mathbb{F}_p}^1 = \Gamma(\mu_0)$

► Topology:

$$\begin{aligned}\pi_* S_{\mathbb{F}_p}^0 &= \begin{cases} \mathbb{F}_2[\tau_0, \tau_1, \dots] \\ \mathbb{F}_p[\tau_0, \dots, \xi_1, \dots] / \tau_i^2 \end{cases} \\ \pi_* S_{\mathbb{F}_p}^1 &= \mathbb{F}_p[\mu_0]\end{aligned}$$

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\mu_0]$$

$\pi_* S_{\mathbb{M}\mathbb{F}_p}^0[\tau^{-1}] = \text{usual dual Steenrod alg. } [\tau^{\pm 1}]$
 $\pi_* S_{\mathbb{M}\mathbb{F}_p}^0 / \tau = \text{usual odd primary dual Steenrod alg.}$

Top
Alg.

i.e., $\pi_* S_{M_{\mathbb{F}_p}}^0 / \tau = \mathbb{F}_p[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots] / \tau_i^2$ also if $p = 2$!

1. What is motivic Hochschild homology with τ inverted? k alg. closed field, $\text{char} \neq p$

Top

$$\pi_* S_{M\mathbb{F}_p[\tau^{-1}]}^0 = \pi_* S_{M\mathbb{F}_p}^0[\tau^{-1}] = \text{usual Steenrod Algebra}[\tau^{\pm 1}]$$

$$\text{Tor}^{\pi_* S_{M\mathbb{F}_p[\tau^{-1}]}^0}(\pi_* M\mathbb{F}_p[\tau^{-1}], \pi_* M\mathbb{F}_p[\tau^{-1}]) \Rightarrow \pi_* S_{M\mathbb{F}_p[\tau^{-1}]}^1$$

is usual $\text{TorSS}[\tau^{\pm 1}]$ and (as for Bö) *but simpler since we get all diff'ls from the extensions*

$$\begin{aligned} \pi_* S_{\mathbb{F}_p[\tau^{-1}]}^1 &= \pi_* S_{\mathbb{F}_p}^1[\tau^{-1}] \\ &= \mathbb{F}_2[\tau^{\pm 1}, \mu_0, \mu_1, \dots] / \mu_i^p - \tau^{p-1} \mu_{i+1} \\ &\cong \mathbb{F}_p[\tau^{\pm 1}, \mu_0] \end{aligned}$$

$$\mu_i = \sigma \tau_i, |\mu_i| = (2p^i, p^i - 1).$$

For comparison:

► Algebra: $\pi_* S_{\mathbb{F}_p}^0 = E(\tau_0),$
 $\pi_* S_{\mathbb{F}_p}^1 = \Gamma(\mu_0)$

► Topology:

$$\pi_* S_{\mathbb{F}_p}^0 = \begin{cases} \mathbb{F}_2[\tau_0, \tau_1, \dots] \\ \mathbb{F}_p[\tau_0, \dots, \xi_1, \dots] / \tau_i^2 \end{cases}$$

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\mu_0]$$

$$S^{i,j} = S^{i-j} \wedge \mathbb{G}_m^{\wedge j}$$

$$\pi_{i,j} X = [S^{i,j}, X]$$

$$\pi_*(M\mathbb{F}_p) = \mathbb{F}_p[\tau],$$

$$|\tau| = (0, -1)$$

$$\pi_* S_{M\mathbb{F}_p}^0[\tau^{-1}] = \pi_* S_{H\mathbb{F}_p}^0[\tau^{\pm 1}]$$

What is motivic Hochschild homology mod τ ? k alg. closed field, $\text{char} \neq 2$

Alg.

$p \neq 2$

$$\begin{aligned} E^2 &= \text{Tor}^{\pi_* S_{M\mathbb{F}_2}^0 / \tau}(\pi_* M\mathbb{F}_2 / \tau, \pi_* M\mathbb{F}_2 / \tau) \\ &= \text{Tor}^{\mathbb{F}_2[\tau_0, \tau_1, \dots, \xi_1, \dots] / \tau_i^2}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_* S_{\mathbb{F}_2}^1 / \tau \end{aligned}$$

Just as the topological odd case

$$E^2 = \bigotimes_{i \geq 0} \Gamma(\bar{\mu}_i) \otimes E(\bar{\lambda}_{i+1}),$$

but – **contrary to the topological situation** –

weight considerations exclude differentials and extensions!

$$\pi_* S_{\mathbb{F}_2}^1 / \tau = \bigotimes_{i \geq 0} \Gamma(\bar{\mu}_i) \otimes E(\bar{\lambda}_{i+1}).$$

Fully algebraic: $(S/\tau\text{-modules}) \simeq (\mathbf{BP}_* \mathbf{BP}\text{-comodules})$

Topology: $\pi_* H\mathbb{F}_p = \mathbb{F}_p$,

$$\pi_* S_{H\mathbb{F}_p}^0 = \begin{cases} \mathbb{F}_2[\tau_0, \tau_1, \dots] \\ \mathbb{F}_p[\tau_0, \dots, \xi_1, \dots] / \tau_i^2 \end{cases}$$

$$\pi_* S_{H\mathbb{F}_p}^1 = \mathbb{F}_p[\mu_0]$$

Motivic: $\pi_*(M\mathbb{F}_p) = \mathbb{F}_p[\tau]$,

$$\pi_{**} S_{M\mathbb{F}_p}^0 = \begin{cases} \mathbb{F}_2[\tau, \tau_0, \dots, \xi_1, \dots] / \tau_i^2 - \tau \xi_{i+1} \\ \mathbb{F}_p[\tau, \tau_0, \dots, \xi_1, \dots] / \tau_i^2 \end{cases}$$

$$|\tau| = (0, -1),$$

$$|\tau_i| = (2 \cdot p^i - 1, p^i - 1),$$

$$|\xi_i| = (2 \cdot p^i - 2, p^i - 1).$$

$$\pi_* S_{M\mathbb{F}_p}^0[\tau^{-1}] = \pi_* S_{H\mathbb{F}_p}^0[\tau^{\pm 1}]$$

$$\pi_* S_{M\mathbb{F}_p}^1[\tau^{-1}] = \mathbb{F}_p[\tau^{\pm 1}, \mu_0]$$

$$\pi_* S_{M\mathbb{F}_p}^0 / \tau = \mathbb{F}_p[\tau_0, \dots, \xi_1, \dots] / \tau_i^2$$

Assembly 1 k alg. closed field, $\text{char} \neq 2$

$\rho = 2$

$$\begin{array}{c}
 \dots \longrightarrow \pi_{i+1,j} S_{\mathbb{F}_2}^1 / \tau \quad \begin{array}{l} \bar{\mu}_i \\ \bar{\lambda}_{i+1} \end{array} \quad \begin{array}{l} (\tau_i) \\ (\xi_{i+1}) \end{array} \\
 \downarrow \partial \\
 \pi_{i,j+1} S_{\mathbb{F}_2}^1 \xrightarrow{\tau} \pi_{i,j} S_{\mathbb{F}_2}^1 \xrightarrow{q} \pi_{i,j} (S_{\mathbb{F}_2}^1 / \tau) \\
 \downarrow \partial \\
 \dots
 \end{array}$$

Handwritten notes: $\bar{\alpha}^2 = 0$, $\bar{\alpha}$, $\downarrow q$, $\pi_{i,j+1} S_{\mathbb{F}_2}^1 / \tau$

Bockstein $\bar{\partial} = q\partial: \pi_{i,j+1} S_{\mathbb{F}_2}^1 / \tau \rightarrow \pi_{i,j} S_{\mathbb{F}_2}^1 / \tau$

$$\bar{\partial} \gamma_{j+2} \bar{\mu}_i = \bar{\lambda}_{i+1} \gamma_j \bar{\mu}_i, \quad \bar{\partial} \bar{\lambda}_{i+1} = 0.$$

obtained from E^1 of the *integral* BarSS. Calculate:

$$H_*(\pi_* S_{\mathbb{F}_2}^1 / \tau, \bar{\partial}) = \bigotimes_{i \geq 0} E(\bar{\mu}_i)$$

Note: $\bar{\mu}_i = q\mu_i$, where μ_i were exactly the classes that survived τ -inversion!

$$\pi_*(M\mathbb{F}_2) = \mathbb{F}_2[\tau], \quad |\tau| = (0, -1)$$

$$\pi_{**} S_{M\mathbb{F}_2}^0 = \mathbb{F}_2[\tau, \tau_0, \dots, \xi_1, \dots] / \tau_i^2 - \tau \xi_{i+1}$$

$$|\tau_i| = (2 \cdot 2^i - 1, 2^i - 1),$$

$$|\xi_i| = (2 \cdot 2^i - 2, 2^i - 1).$$

$$\pi_* S_{M\mathbb{F}_2}^0[\tau^{-1}] = \mathbb{F}_2[\tau^{\pm 1}, \tau_0, \tau_1, \dots]$$

$$\pi_* S_{M\mathbb{F}_2}^0 / \tau = \mathbb{F}_2[\tau_0, \dots, \xi_1, \dots] / \tau_i^2$$

$$\pi_* S_{M\mathbb{F}_2}^1[\tau^{-1}] = \mathbb{F}_2[\tau^{\pm 1}, \mu_0]$$

$$\pi_* S_{M\mathbb{F}_2}^1 / \tau = \bigotimes_{i \geq 0} \Gamma(\bar{\mu}_i) \otimes E(\bar{\lambda}_{i+1})$$

NB! exceedingly convenient!!

Assembly 2 k alg. closed field, $\text{char} \neq 2$

When $p=2$, the d' -differentials in the Integral Bockstein are messy - the reason to do it this way.

When p is odd the d' 's are comfortably spread out (cf. d'' 's)

This pins down the motivic Hochschild homology:

- ▶ τ -torsion is not τ -divisible,
- ▶ non-torsion part assembles to

$$\mathbb{F}_2[\tau, \mu_0, \mu_1 \dots] / \mu_i^2 - \tau \mu_{i+1},$$

where $q\mu_j = \bar{\mu}_j$

- ▶ τ -torsion = the image of $\partial: \pi_{i+1,j} S_{M\mathbb{F}_2}^1 / \tau \rightarrow \pi_{i,j+1} S_{M\mathbb{F}_2}^1$
maps injectively by $q: \pi_{i,j} S_{M\mathbb{F}_2}^1 \rightarrow \pi_{i,j} S_{M\mathbb{F}_2}^1 / \tau$.

$$\pi_* S_{M\mathbb{F}_2}^1[\tau^{-1}] = \mathbb{F}_2[\tau^{\pm 1}, \mu_0]$$

$$\pi_* S_{M\mathbb{F}_2}^1 / \tau = \bigotimes_{i \geq 0} \Gamma(\bar{\mu}_i) \otimes E(\bar{\lambda}_{i+1})$$

Bockstein

$$\bar{\partial} = q\partial: \pi_{i,j+1} S_M^1 / \tau \rightarrow \pi_{i,j} S_M^1 / \tau$$

$$\bar{\partial} \gamma_{j+2} \bar{\mu}_i = \bar{\lambda}_{i+1} \gamma_j \bar{\mu}_i, \quad \bar{\partial} \bar{\lambda}_{i+1} = 0.$$

$$H_*(\pi_* S_M^1 / \tau, \bar{\partial}) = \bigotimes_{i \geq 0} E(\bar{\mu}_i)$$

Assembly 3 k alg. closed field, $\text{char} \neq 2$

All that's left to do is to figure out the multiplicative structure of the τ -torsion: *+ describe in $\bar{\mathcal{J}}$.*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (\tau\text{-torsion}) & \longrightarrow & \pi_* S^1_{M\mathbb{F}_2} & \longrightarrow & \mathbb{F}_2[\tau, \mu_0, \mu_1, \dots] / \mu_i^2 - \tau\mu_{i+1} \longrightarrow 0 \\
 & & \cong \downarrow q & & \downarrow q & & \\
 0 & \longrightarrow & \text{im } \bar{\partial} & \longrightarrow & \bigotimes_{i \geq 0} \Gamma(\bar{\mu}_i) \otimes E(\bar{\lambda}_{i+1}) & &
 \end{array}$$

culminating in a pullback of commutative $\mathbb{F}_2[\tau]$ -algebras

$$\begin{array}{ccc}
 \pi_* S^1_{\mathbb{F}_2} & \longrightarrow & \mathbb{F}_2[\tau, \mu_i] / (\mu_i^2 - \tau\mu_{i+1}) \\
 \downarrow & & \downarrow \\
 \mathbb{F}_2[\bar{\mu}_i, x_{S,f}] / \mathcal{J} & \longrightarrow & \mathbb{F}_2[\bar{\mu}_i] / \bar{\mu}_i^2
 \end{array}$$

$$\pi_* S^1_{M\mathbb{F}_2}[\tau^{-1}] = \mathbb{F}_2[\tau^{\pm 1}, \mu_0]$$

$$\pi_* S^1_{M\mathbb{F}_2} / \tau = \bigotimes_{i \geq 0} \Gamma(\bar{\mu}_i) \otimes E(\bar{\lambda}_{i+1})$$

Bockstein

$$\bar{\partial} = q\partial: \pi_{i,j+1} S^1_M / \tau \rightarrow \pi_{i,j} S^1_M / \tau$$

$$\mathcal{J} = (\bar{\mu}_i^2, \quad x_{S,f} \cdot x_{T,g} - \sum_{u \in \text{supp}(f+g) - S \cup T} \epsilon_u \cdot x_{S \cup T \cup \{u\}, f+g}).$$

$$\bar{\partial} \gamma_{j+2} \bar{\mu}_i = \bar{\lambda}_{i+1} \gamma_j \bar{\mu}_i, \quad \bar{\partial} \bar{\lambda}_{i+1} = 0.$$

$$H_*(\pi_* S^1_M / \tau, \bar{\partial}) = \bigotimes_{i \geq 0} E(\bar{\mu}_i) \quad \cong$$

Summing up k alg. closed field, $\text{char} \neq p$

МНН. Пр.

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\tau][\mu_i, x_{S,f}]/\mathcal{I}$$

$$\tau^{p-1} \mu_{i+1} = \mu_i^p$$

$$\tau^{p-1} x_{S,f} = 0$$

$$x_{S,f} \cdot x_{T,g} = \sum_u \epsilon_u \cdot x_{S \cup T \cup \{u\}, f+g}.$$

For the curious:

$$x_{S,f} = \partial \left(\prod_{m \in S} \bar{\lambda}_{m+1} \gamma_{pf(m)-p} \bar{\mu}_m \right) \left(\prod_{n \notin S} \gamma_{pf(n)} \bar{\mu}_n \right)$$

f governs the divided power exponents

S detects the presence of exterior classes

$$\epsilon_u = \epsilon_{u,S,T,f,g} = K_{S \cup \{u\}, T \cup \{\min \text{supp}(f+g)\}, f,g} + K_{S \cup \{\min \text{supp}(f+g)\}, T \cup \{u\}, f,g} \in \mathbb{F}_p$$

where

$$K_{S,T,f,g} = \left(\prod_{s \in S} \binom{f_s - 1 + g_s}{f_s - 1} \right) \left(\prod_{t \in T} \binom{f_t + g_t - 1}{f_t} \right) \left(\prod_{c \notin S \cup T} \binom{f_c + g_c}{f_c} \right)$$

very often zero...



Immediate consequences:

1. Torsion phenomena (MHH not free ...)
2. Fixed points/Tate display extra τ -divisibility

Some projects:

1. Other rings than $\mathbb{F}_p$ and to ground fields that are not alg. closed (cf equivariant stable htt)
2. Develop a trace from K-theory – will it be any good?
3. Arithmetic of commutative rings in motivic setting: what is the importance of the cotangent complex? / smoothness? / HKR?...
4. Chromatic/equivariant properties: how does the τ^{p-1} -divided power structure alter periodicity?
5. Tensoring over $\mathbb{G}_m$ instead of S^1 .
6. What hidden relations in topology are captured motivically?

Motivic Hochschild homology:

$$\pi_* S_{\mathbb{F}_p}^1 = \mathbb{F}_p[\tau][\mu_i, x_{S,f}]/\mathcal{I}$$

- ▶ $\tau^{p-1} \mu_{i+1} = \mu_i^p$
- ▶ $\tau^{p-1} x_{S,f} = 0$
- ▶ $x_{S,f} \cdot x_{T,g} = \sum_u \epsilon_u \cdot x_{S \cup T \cup \{u\}, f+g}$

k alg. closed field, $\text{char} \neq p$