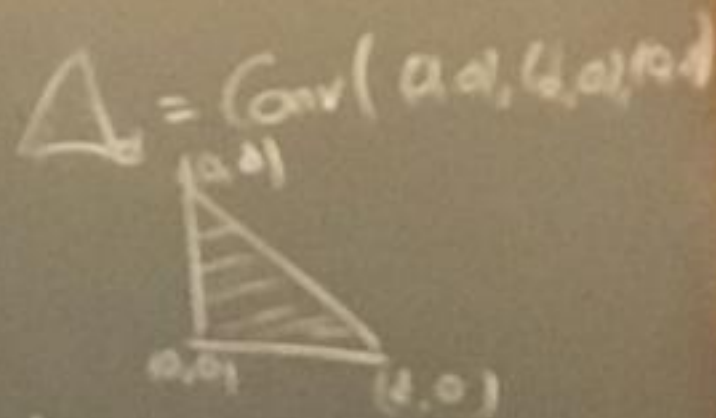


The combinatorial structure of a tropical curve by its dual subdivision

↳ subdivision of



Tropical curve

vertex

edges

conn comp
of $\mathbb{R}^2 \setminus C$

Dual subdivision

max cells

edges

vertex

- dual edges are orthogonal
- inclusions are inverted

$$C = C_1 \cup C_2$$

$p \in C_1 \cap C_2$

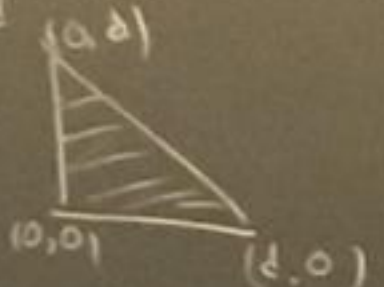
parallelogram P

† Def. $\text{mult}_p(C_1, C_2) := \text{Area } P$

⇒ Proof of Bézout's thm for tropical curves (B. Sturmfels)

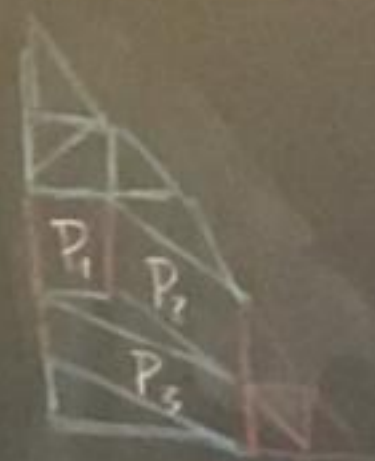
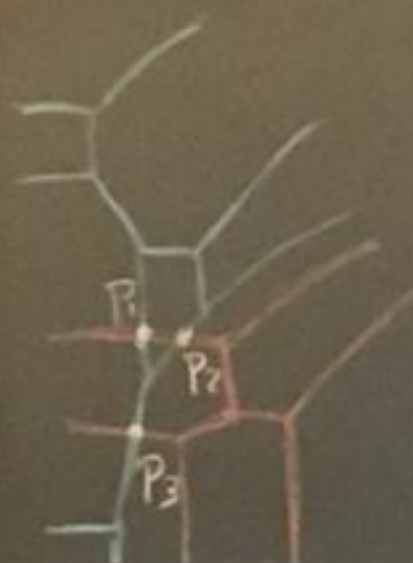
$$\text{Area } \Delta_{d_1+d_2} - \text{Area } \Delta_{d_1} - \text{Area } \Delta_{d_2} = d_1 d_2$$

al

$$C = \text{Conv}((0,0), (d,0), (a,d))$$


es
gonal
are inverted

al curves
(B. Sturmfels)



$$\begin{aligned} \text{mult}_{P_1}(C_1, C_2) &= 1 \\ \text{mult}_{P_2} &= 2 \\ \text{mult}_{P_3} &= 3 \\ \sum &= 6 \end{aligned}$$

Bézout for curves enriched in $\text{GW}(k)$ (S McKean)

k a field
 $C_1 = V(F_1)$ $C_2 = V(F_2) \in \mathbb{P}_k^2$ st $d_1 + d_2$ odd
 $\uparrow \text{deg } d_1$ $\uparrow \text{deg } d_2$

Then $\sum_{x \in C_1 \cap C_2} \text{Tr}_{k^m/k} \langle \det \text{Jac}(F_1, F_2)(x) \rangle = \frac{d_1 d_2}{2} \quad \forall F_i \in \text{GW}(k)$
 $\uparrow$ hyperbolic form

Field of Puiseux Series $k\{\{t\}\}$
 $k\{\{t\}\} = \left\{ a = \sum_{i=0}^{\infty} a_i t^{i/n} : a_i \in k, n \in \mathbb{N} \right\}$

Q: What is $GW(k\{\{t\}\})$?

A: $GW(F)$ is generated by $\langle a \rangle$ $a \in F^x / (F^x)^2$

Exercise $a = \sum_{i=0}^{\infty} a_i t^{i/n}$ $\xrightarrow{(k\{\{t\}\})^x / (k\{\{t\}\})^2} a_{i_0} / (k^x)^2$

(S McKean)

odd

$H \in GW(k)$

↑ hyperbolic form

$$F(x, y) = a(t) + b(t)x + c(t)y \in k\{\{t\}\}[x, y]$$

$$x(t) = x_0 t^{-z_0} + h_0 t^{e_0} + \dots$$

$$y(t) = y_0 t^{-w_0} + h_0 t^{f_0} + \dots$$

$$a(t) = a_0 t^d + h_0 t$$

$$b(t) = b_0 t^e + h_0 t$$

$$c(t) = c_0 t^f + h_0 t$$

Want to solve $F(x(t), y(t)) = 0$

$$a_0 t^d + h_0 t + b_0 t^{e-z_0} x_0 + h_0 t + c_0 t^{f-w_0} y_0 + h_0 t$$

Need $\min(d, e+z_0, f+w_0)$ is at least attained twice

More generally,

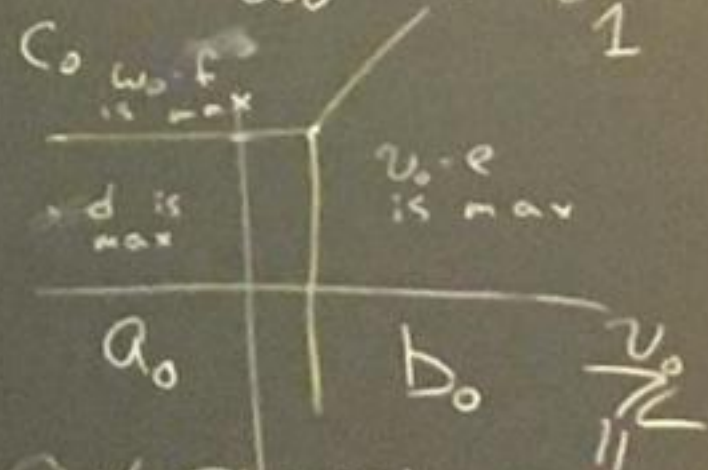


In general, $F \in k\{\{t\}\}[x, y]$
 of deg d
 $\leadsto$ tropical curve of degree d

$k = \mathbb{C}$, $F_1, F_2 \in k\{\{t\}\}[x, y]$
 $\leadsto 2$ trop curves C_1, C_2 $p \in C_1 \cap C_2$

$\text{mult}_p(C_1, C_2) = \# \text{zeros } (x(t), y(t)) \text{ of}$
 $F_1 \& F_2 \text{ with } (-v_0, -w_0) = p$

$$\begin{aligned} d=1 & \quad v_0 = e - d = 1 \\ e=2 & \quad w_0 = f - d = 2 \\ f=3 & \quad w_0 = v_0 + \underbrace{f - e}_1 \end{aligned}$$



$$\text{GW}(\mathbb{P}^2\{\{t\}\}) = \text{GW}(\mathbb{P}^2)$$

Q: What is $\text{Tr}_{E/k^{\text{alg}}}$ ($\langle \det \text{Jac}(F_1, F_2)(x(t), y(t)) \rangle$)
 k arbitrary $\uparrow$ coord ring of $(x(t), y(t))$ with $(-v_0, -w_0) = P$

Need to remember the initials of the coeff of F_1 & F_2

Def An enriched tropical curve $\hat{C} = (C, (\alpha_I))$
 is a tropical curve together $\uparrow$ trop curve $\uparrow$ coeffs of $k^x / (k^x)^2$

with a coefficient α_I assigned to each comp of $\mathbb{R}^2 \setminus C$ / vertex in dual subdivision

Def Call $I = (i_1, i_2) \in \mathbb{Z}^2$ odd if i_1 and i_2 are both odd.

Thm (Jaramillo Puentes - P)

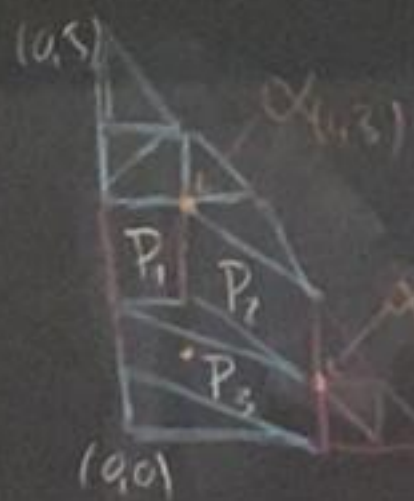
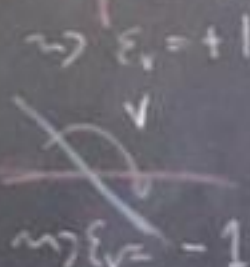
$$Tr_{E/k(x)} \langle \det Jac(T_1, T_2)(x(t), y(t)) \rangle$$

$$= \sum_{\substack{v \text{ odd} \\ \text{vertex} \\ \text{of parallelogram} \\ \text{dual to } p}} \langle \epsilon_v \alpha_v \rangle$$

+ hyperbolic

$\alpha_v = \text{coeff of } v$
 $\epsilon_v = \text{Sign}$

$$p \in C_1 \cap C_2$$



$$\begin{aligned} \tilde{\text{mult}}_{p_1}(\tilde{C}_1, \tilde{C}_2) &= \langle -\alpha_{(1,3)} \rangle \\ \tilde{\text{mult}}_{p_2} &= \langle \alpha_{(0,3)} \rangle + \langle \alpha_{(3,1)} \rangle \\ \tilde{\text{mult}}_{p_3} &= \langle -\alpha_{(3,1)} \rangle + H \end{aligned}$$

$$\sum = 3H$$

Cor ^{quadr enriched} Bézout in $GL_2(k(t))$
in case $d_1 + d_2$ odd

Why $d_1 + d_2$ odd? Then all odd points lie in
the interior of $\Delta_{d_1+d_2}$

Generalizations

- can also define 'banded tropical hyperbodies'
→ Bézout not only for curves
- Can compute intersections in toric varieties
- Can study the non-relativity analogue case